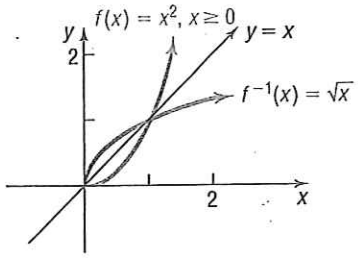
Figure 17



STEP 1: In the equation $y = x^2, x \geq 0$, interchange the variables x and y. The result is

$$x = y^2 \qquad y \geq 0$$

This equation defines (implicitly) the inverse function.

STEP 2: Solve for y to get the explicit form of the inverse. Since $y \geq 0$, only one solution for y is obtained: $y = \sqrt{x}$. So $f^{-1}(x) = \sqrt{x}$.

STEP 3: Check: $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x$ since $x \geq 0$

$$f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

Figure 17 illustrates the graphs of $f(x) = x^2, x \geq 0$, and $f^{-1}(x) = \sqrt{x}$. Note that the domain of f = range of $f^{-1} = [0, \infty)$ and the domain of f^{-1} = range of $f = [0, \infty)$.

SUMMARY

1. If a function f is one-to-one, then it has an inverse function f^{-1}.
2. Domain of f = Range of f^{-1}; Range of f = Domain of f^{-1}.
3. To verify that f^{-1} is the inverse of f, show that $f^{-1}(f(x)) = x$ for every x in the domain of f and that $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.
4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

4.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Is the set of ordered pairs $\{(1, 3), (2, 3), (-1, 2)\}$ a function? Why or why not? (pp. 45–46)
2. Where is the function $f(x) = x^2$ increasing? Where is it decreasing? (p. 68)
3. What is the domain of $f(x) = \dfrac{x + 5}{x^2 + 3x - 18}$? (pp. 49–51)
4. Simplify: $\dfrac{\frac{1}{x} + 1}{\frac{1}{x^2} - 1}$ (pp. A50–A52)

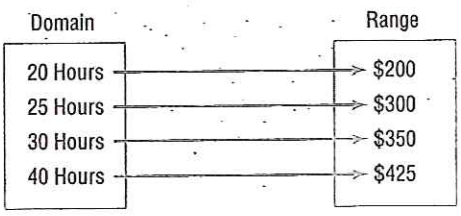
Concepts and Vocabulary

5. If x_1 and x_2 are any two different inputs of a function f, then f is one-to-one if _____.
6. If every horizontal line intersects the graph of a function f at no more than one point, f is a(n) _____ function.
7. If f is a one-to-one function and $f(3) = 8$, then $f^{-1}(8) =$ _____.
8. If f^{-1} denotes the inverse of a function f, then the graphs of f and f^{-1} are symmetric with respect to the line _____.
9. If the domain of a one-to-one function f is $[4, \infty)$, then the range of its inverse, f^{-1}, is _____.
10. ***True or False*** If f and g are inverse functions, the domain of f is the same as the range of g.

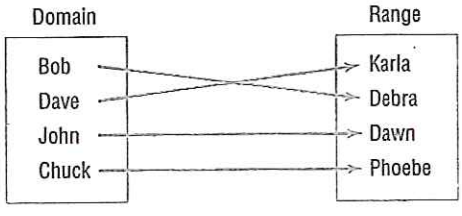
Skill Building

In Problems 11–18, determine whether the function is one-to-one.

11.



12.



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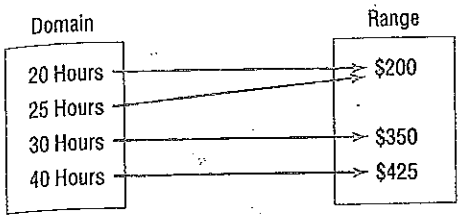

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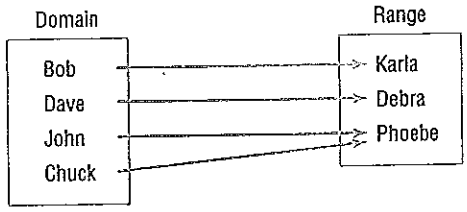


15. $\{(2,6), (-3,6), (4,9), (1,10)\}$

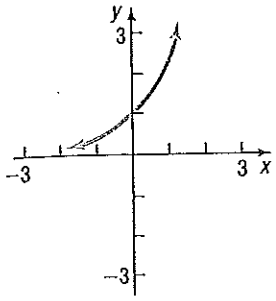
16. $\{(-2,5), (-1,3), (3,7), (4,12)\}$

17. $\{(0,0), (1,1), (2,16), (3,81)\}$

18. $\{(1,2), (2,8), (3,18), (4,32)\}$

In Problems 19–24, the graph of a function f is given. Use the horizontal-line test to determine whether f is one-to-one.

19.



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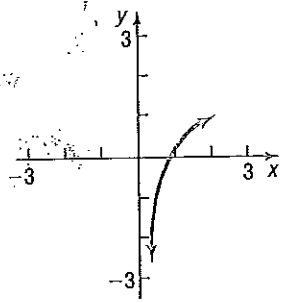

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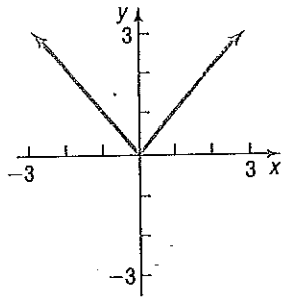

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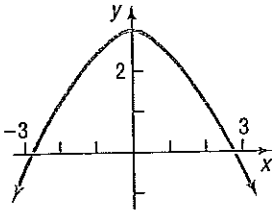

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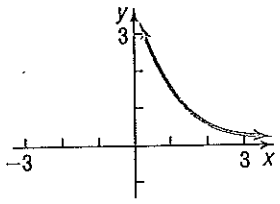

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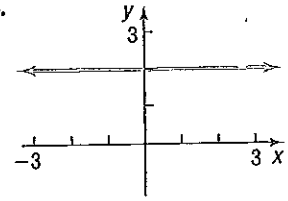
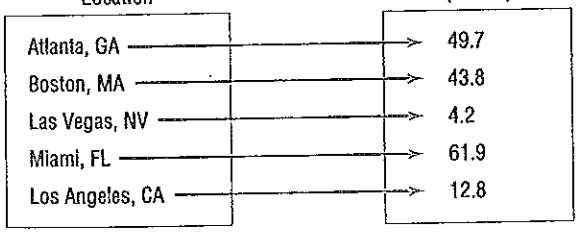


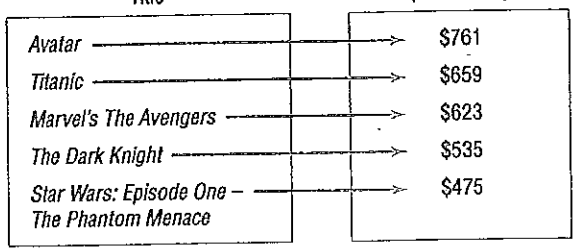
In Problems 25–32, find the inverse of each one-to-one function. State the domain and the range of each inverse function.

25.



Source: currentresults.com

26.

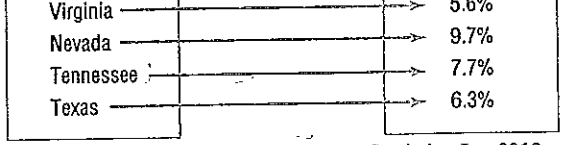


Source: boxofficemojo.com

27.

Source: acequotes.com

28.



Source: United States Bureau of Labor Statistics, Jan. 2013

29. $\{(-3,5), (-2,9), (-1,2), (0,11), (1,-5)\}$

30. $\{(-2,2), (-1,6), (0,8), (1,-3), (2,9)\}$

31. $\{(-2,1), (-3,2), (-10,0), (1,9), (2,4)\}$

32. $\{(-2,-8), (-1,-1), (0,0), (1,1), (2,8)\}$

In Problems 33–42, verify that the functions f and g are inverses of each other by showing that $f(g(x)) = x$ and $g(f(x)) = x$. Give any values of x that need to be excluded from the domain of f and the domain of g.

33. $f(x) = 3x + 4; \quad g(x) = \frac{1}{3}(x - 4)$

34. $f(x) = 3 - 2x; \quad g(x) = -\frac{1}{2}(x - 3)$

35. $f(x) = 4x - 8; \quad g(x) = \frac{x}{4} + 2$

36. $f(x) = 2x + 6; \quad g(x) = \frac{1}{2}x - 3$

37. $f(x) = x^3 - 8$; $g(x) = \sqrt[3]{x + 8}$

38. $f(x) = (x - 2)^2, x \geq 2$; $g(x) = \sqrt{x} + 2$

39. $f(x) = \frac{1}{x}$; $g(x) = \frac{1}{x}$

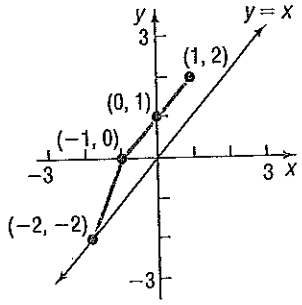
40. $f(x) = x$; $g(x) = x$

41. $f(x) = \frac{2x + 3}{x + 4}$; $g(x) = \frac{4x - 3}{2 - x}$

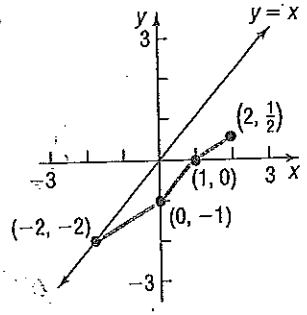
42. $f(x) = \frac{x - 5}{2x + 3}$; $g(x) = \frac{3x + 5}{1 - 2x}$

In Problems 43–48, the graph of a one-to-one function f is given. Draw the graph of the inverse function f^{-1}. For convenience (and as a hint), the graph of $y = x$ is also given.

43.



44.



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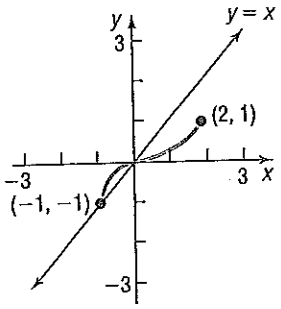

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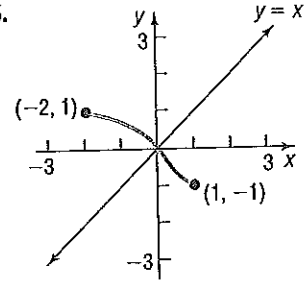

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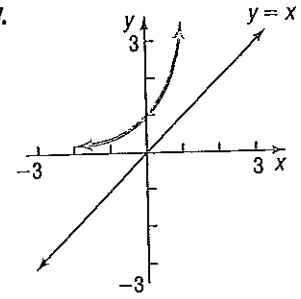
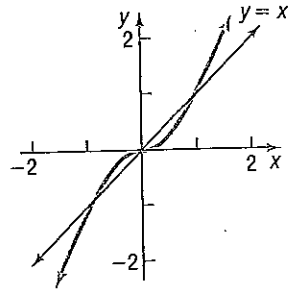


48.



In Problems 49–60, the function f is one-to-one. Find its inverse and check your answer. Graph f, f^{-1}, and $y = x$ on the same coordinate axes.

49. $f(x) = 3x$

50. $f(x) = -4x$

51. $f(x) = 4x + 2$

52. $f(x) = 1 - 3x$

53. $f(x) = x^3 - 1$

54. $f(x) = x^3 + 1$

55. $f(x) = x^2 + 4, \quad x \geq 0$

56. $f(x) = x^2 + 9, \quad x \geq 0$

57. $f(x) = \frac{4}{x}$

58. $f(x) = -\frac{3}{x}$

59. $f(x) = \frac{1}{x - 2}$

60. $f(x) = \frac{4}{x + 2}$

In Problems 61–72, the function f is one-to-one. Find its inverse and check your answer.

61. $f(x) = \frac{2}{3 + x}$

62. $f(x) = \frac{4}{2 - x}$

63. $f(x) = \frac{3x}{x + 2}$

64. $f(x) = -\frac{2x}{x - 1}$

65. $f(x) = \frac{2x}{3x - 1}$

66. $f(x) = -\frac{3x + 1}{x}$

67. $f(x) = \frac{3x + 4}{2x - 3}$

68. $f(x) = \frac{2x - 3}{x + 4}$

69. $f(x) = \frac{2x + 3}{x + 2}$

70. $f(x) = \frac{-3x - 4}{x - 2}$

71. $f(x) = \frac{x^2 - 4}{2x^2}, \quad x > 0$

72. $f(x) = \frac{x^2 + 3}{3x^2}, \quad x > 0$

Applications and Extensions

73. Use the graph of $y = f(x)$ given in Problem 43 to evaluate the following:
(a) $f(-1)$ (b) $f(1)$ (c) $f^{-1}(1)$ (d) $f^{-1}(2)$

74. Use the graph of $y = f(x)$ given in Problem 44 to evaluate the following:
(a) $f(2)$ (b) $f(1)$ (c) $f^{-1}(0)$ (d) $f^{-1}(-1)$

75. If $f(7) = 13$ and f is one-to-one, what is $f^{-1}(13)$?

76. If $g(-5) = 3$ and g is one-to-one, what is $g^{-1}(3)$?

77. The domain of a one-to-one function f is $[5, \infty)$, and its range is $[-2, \infty)$. State the domain and the range of f^{-1}.

78. The domain of a one-to-one function f is $[0, \infty)$, and its range is $[5, \infty)$. State the domain and the range of f^{-1}.

79. The domain of a one-to-one function g is $(-\infty, 0]$, and its range is $[0, \infty)$. State the domain and the range of g^{-1}.

80. The domain of a one-to-one function g is $[0, 15]$, and its range is $(0, 8)$. State the domain and the range of g^{-1}.

81. A function $y = f(x)$ is increasing on the interval $(0, 5)$. What conclusions can you draw about the graph of $y = f^{-1}(x)$?

82. A function $y = f(x)$ is decreasing on the interval $(0, 5)$. What conclusions can you draw about the graph of $y = f^{-1}(x)$?

83. Find the inverse of the linear function

$$f(x) = mx + b, \quad m \neq 0$$

84. Find the inverse of the function

$$f(x) = \sqrt{r^2 - x^2}, \quad 0 \leq x \leq r$$

85. A function f has an inverse function. If the graph of f lies in quadrant I, in which quadrant does the graph of f^{-1} lie?

86. A function f has an inverse function. If the graph of f lies in quadrant II, in which quadrant does the graph of f^{-1} lie?

87. The function $f(x) = |x|$ is not one-to-one. Find a suitable restriction on the domain of f so that the new function that results is one-to-one. Then find the inverse of f.

88. The function $f(x) = x^4$ is not one-to-one. Find a suitable restriction on the domain of f so that the new function that results is one-to-one. Then find the inverse of f.

In applications, the symbols used for the independent and dependent variables are often based on common usage. So, rather than using $y = f(x)$ to represent a function, an applied problem might use $C = C(q)$ to represent the cost C of manufacturing q units of a good since, in economics, q is used for output. Because of this, the inverse notation f^{-1} used in a pure mathematics problem is not used when finding inverses of applied problems. Rather, the inverse of a function such as $C = C(q)$ will be $q = q(C)$. So $C = C(q)$ is a function that represents the cost C as a function of the output q, and $q = q(C)$ is a function that represents the output q as a function of the cost C. Problems 89–92 illustrate this idea.

89. **Vehicle Stopping Distance** Taking into account reaction time, the distance d (in feet) that a car requires to come to a complete stop while traveling r miles per hour is given by the function

$$d(r) = 6.97r - 90.39$$

(a) Express the speed r at which the car is traveling as a function of the distance d required to come to a complete stop.
(b) Verify that $r = r(d)$ is the inverse of $d = d(r)$ by showing that $r(d(r)) = r$ and $d(r(d)) = d$.
(c) Predict the speed that a car was traveling if the distance required to stop was 300 feet.

90. **Height and Head Circumference** The head circumference C of a child is related to the height H of the child (both in inches) through the function

$$H(C) = 2.15C - 10.53$$

(a) Express the head circumference C as a function of height H.
(b) Verify that $C = C(H)$ is the inverse of $H = H(C)$ by showing that $H(C(H)) = H$ and $C(H(C)) = C$.
(c) Predict the head circumference of a child who is 26 inches tall.

91. **Ideal Body Weight** One model for the ideal body weight W for men (in kilograms) as a function of height h (in inches) is given by the function

$$W(h) = 50 + 2.3(h - 60)$$

(a) What is the ideal weight of a 6-foot male?
(b) Express the height h as a function of weight W.
(c) Verify that $h = h(W)$ is the inverse of $W = W(h)$ by showing that $h(W(h)) = h$ and $W(h(W)) = W$.
(d) What is the height of a male who is at his ideal weight of 80 kilograms?

[Note: The ideal body weight W for women (in kilograms) as a function of height h (in inches) is given by $W(h) = 45.5 + 2.3(h - 60)$.]

92. **Temperature Conversion** The function $F(C) = \frac{9}{5}C + 32$ converts a temperature from C degrees Celsius to F degrees Fahrenheit.

(a) Express the temperature in degrees Celsius C as a function of the temperature in degrees Fahrenheit F.
(b) Verify that $C = C(F)$ is the inverse of $F = F(C)$ by showing that $C(F(C)) = C$ and $F(C(F)) = F$.
(c) What is the temperature in degrees Celsius if it is 70 degrees Fahrenheit?

93. **Income Taxes** The function

$$T(g) = 4991.25 + 0.25(g - 36{,}250)$$

represents the 2013 federal income tax T (in dollars) due for a "single" filer whose modified adjusted gross income is g dollars, where $36{,}250 \leq g \leq 87{,}850$.
(a) What is the domain of the function T?
(b) Given that the tax due T is an increasing linear function of modified adjusted gross income g, find the range of the function T.
(c) Find adjusted gross income g as a function of federal income tax T. What are the domain and the range of this function?

94. **Income Taxes** The function

$$T(g) = 1785 + 0.15(g - 17{,}850)$$

represents the 2013 federal income tax T (in dollars) due for a "married filing jointly" filer whose modified adjusted gross income is g dollars, where $17{,}850 \leq g \leq 72{,}500$.
(a) What is the domain of the function T?
(b) Given that the tax due T is an increasing linear function of modified adjusted gross income g, find the range of the function T.
(c) Find adjusted gross income g as a function of federal income tax T. What are the domain and the range of this function?

95. **Gravity on Earth** If a rock falls from a height of 100 meters on Earth, the height H (in meters) after t seconds is approximately

$$H(t) = 100 - 4.9t^2$$

(a) In general, quadratic functions are not one-to-one. However, the function H is one-to-one. Why?
(b) Find the inverse of H and verify your result.
(c) How long will it take a rock to fall 80 meters?