SUMMARY

Properties of the Exponential Function

$f(x) = a^x, \quad a > 1$

Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$
x-intercepts: none; y-intercept: 1
Horizontal asymptote: x-axis ($y = 0$) as $x \to -\infty$
Increasing; one-to-one; smooth; continuous
See Figure 22 for a typical graph.

$f(x) = a^x, \quad 0 < a < 1$

Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$
x-intercepts: none; y-intercept: 1
Horizontal asymptote: x-axis ($y = 0$) as $x \to \infty$
Decreasing; one-to-one; smooth; continuous
See Figure 26 for a typical graph.

If $a^u = a^v$, then $u = v$.

4.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. $4^3 =$ ____; $8^{2/3} =$ ____; $3^{-2} =$ ______. (pp. A7–A8 and pp. A58–A59)
2. Solve: $x^2 + 3x = 4$ (pp. A67–A69)
3. ***True or False*** To graph $y = (x - 2)^3$, shift the graph of $y = x^3$ to the left 2 units. (pp. 89–92)
4. Find the average rate of change of $f(x) = 3x - 5$ from $x = 0$ to $x = 4$. (pp. 72–74; 119–122)
5. ***True or False*** The function $f(x) = \dfrac{2x}{x - 3}$ has $y = 2$ as a horizontal asymptote. (pp. 237–240)

Concepts and Vocabulary

6. A(n) ________ ________ is a function of the form $f(x) = Ca^x$, where $a > 0$, $a \neq 1$, and $C \neq 0$ are real numbers. The base a is the ________ and C is the ________.
7. For an exponential function $f(x) = Ca^x$, $\dfrac{f(x+1)}{f(x)} =$ ____.
8. ***True or False*** The domain of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is the set of all real numbers.
9. ***True or False*** The range of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is the set of all real numbers.
10. ***True or False*** The graph of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, has no x-intercept.
11. The graph of every exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, passes through three points: ________, ________, and ________.
12. If the graph of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is decreasing, then a must be less than ____.
13. If $3^x = 3^4$, then $x =$ ____.
14. ***True or False*** The graphs of $y = 3^x$ and $y = \left(\dfrac{1}{3}\right)^x$ are identical.

Skill Building

In Problems 15–24, approximate each number using a calculator. Express your answer rounded to three decimal places.

15. (a) $3^{2.2}$ (b) $3^{2.23}$ (c) $3^{2.236}$ (d) $3^{\sqrt{5}}$
16. (a) $5^{1.7}$ (b) $5^{1.73}$ (c) $5^{1.732}$ (d) $5^{\sqrt{3}}$
17. (a) $2^{3.14}$ (b) $2^{3.141}$ (c) $2^{3.1415}$ (d) 2^{π}
18. (a) $2^{2.7}$ (b) $2^{2.71}$ (c) $2^{2.718}$ (d) 2^{e}
19. (a) $3.1^{2.7}$ (b) $3.14^{2.71}$ (c) $3.141^{2.718}$ (d) π^{e}
20. (a) $2.7^{3.1}$ (b) $2.71^{3.14}$ (c) $2.718^{3.141}$ (d) e^{π}
21. $e^{1.2}$
22. $e^{-1.3}$
23. $e^{-0.85}$
24. $e^{2.1}$

In Problems 25–32, determine whether the given function is linear, exponential, or neither. For those that are linear functions, find a linear function that models the data; for those that are exponential, find an exponential function that models the data.

25.

x	$f(x)$
−1	3
0	6
1	12
2	18
3	30

26.

x	$g(x)$
−1	2
0	5
1	8
2	11
3	14

27.

x	$H(x)$
−1	$\frac{1}{4}$
0	1
1	4
2	16
3	64

28.

x	$f(x)$
−1	$\frac{3}{2}$
0	3
1	6
2	12
3	24

29.

x	$H(x)$
−1	2
0	4
1	6
2	8
3	10

30.

x	$g(x)$
−1	6
0	1
1	0
2	3
3	10

31.

x	$F(x)$
−1	$\frac{2}{3}$
0	1
1	$\frac{3}{2}$
2	$\frac{9}{4}$
3	$\frac{27}{8}$

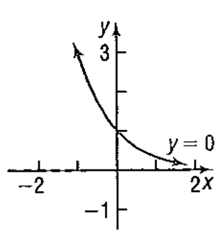
32.

x	$F(x)$
−1	$\frac{1}{2}$
0	$\frac{1}{4}$
1	$\frac{1}{8}$
2	$\frac{1}{16}$
3	$\frac{1}{32}$

In Problems 33–40, the graph of an exponential function is given. Match each graph to one of the following functions.

(a) $y = 3^x$ (b) $y = 3^{-x}$ (c) $y = -3^x$ (d) $y = -3^{-x}$

(e) $y = 3^x - 1$ (f) $y = 3^{x-1}$ (g) $y = 3^{1-x}$ (h) $y = 1 - 3^x$

33.


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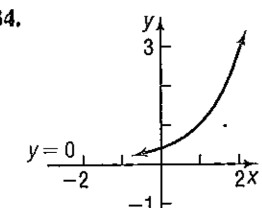

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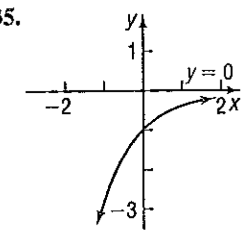

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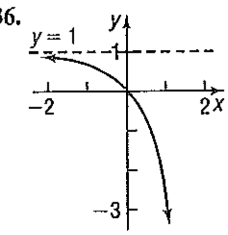

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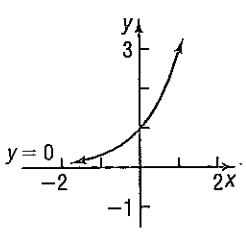

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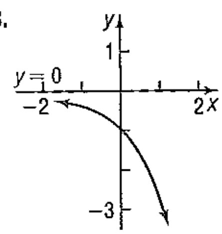

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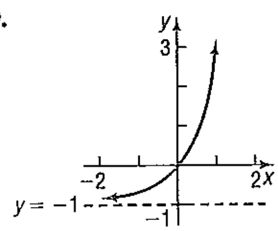

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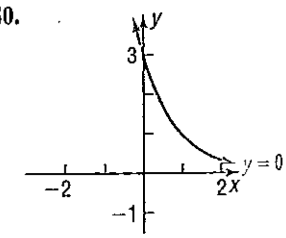


In Problems 41–52, use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

41. $f(x) = 2^x + 1$ 42. $f(x) = 3^x - 2$ 43. $f(x) = 3^{x-1}$ 44. $f(x) = 2^{x+2}$

45. $f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$ 46. $f(x) = 4 \cdot \left(\frac{1}{3}\right)^x$ 47. $f(x) = 3^{-x} - 2$ 48. $f(x) = -3^x + 1$

49. $f(x) = 2 + 4^{x-1}$ 50. $f(x) = 1 - 2^{x+3}$ 51. $f(x) = 2 + 3^{x/2}$ 52. $f(x) = 1 - 2^{-x/3}$

In Problems 53–60, begin with the graph of $y = e^x$ [Figure 28] and use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

53. $f(x) = e^{-x}$ 54. $f(x) = -e^x$ 55. $f(x) = e^{x+2}$ 56. $f(x) = e^x - 1$

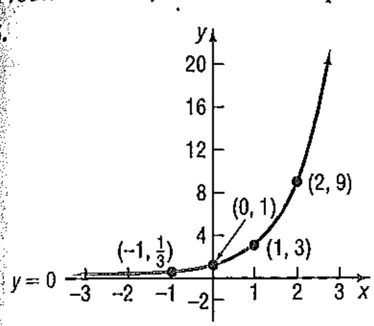
57. $f(x) = 5 - e^{-x}$ 58. $f(x) = 9 - 3e^{-x}$ 59. $f(x) = 2 - e^{-x/2}$ 60. $f(x) = 7 - 3e^{2x}$

In Problems 61–80, solve each equation.

61. $7^x = 7^3$
62. $5^x = 5^{-6}$
63. $2^{-x} = 16$
64. $3^{-x} = 81$
65. $\left(\frac{1}{5}\right)^x = \frac{1}{25}$
66. $\left(\frac{1}{4}\right)^x = \frac{1}{64}$
67. $2^{2x-1} = 4$
68. $5^{x+3} = \frac{1}{5}$
69. $3^{x^3} = 9^x$
70. $4^{x^2} = 2^x$
71. $8^{-x+14} = 16^x$
72. $9^{-x+15} = 27^x$
73. $3^{x^2-7} = 27^{2x}$
74. $5^{x^2+8} = 125^{2x}$
75. $4^x \cdot 2^{x^2} = 16^2$
76. $9^{2x} \cdot 27^{x^2} = 3^{-1}$
77. $e^x = e^{3x+8}$
78. $e^{3x} = e^{2-x}$
79. $e^{x^2} = e^{3x} \cdot \frac{1}{e^2}$
80. $(e^4)^x \cdot e^{x^2} = e^{12}$

81. If $4^x = 7$, what does 4^{-2x} equal?
82. If $2^x = 3$, what does 4^{-x} equal?
83. If $3^{-x} = 2$, what does 3^{2x} equal?
84. If $5^{-x} = 3$, what does 5^{3x} equal?

In Problems 85–88, determine the exponential function whose graph is given.

85.


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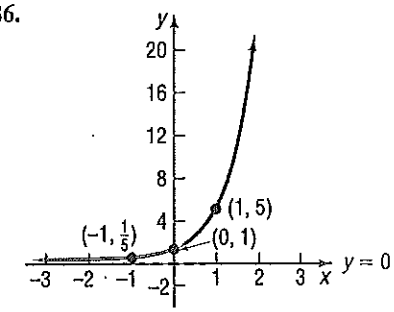

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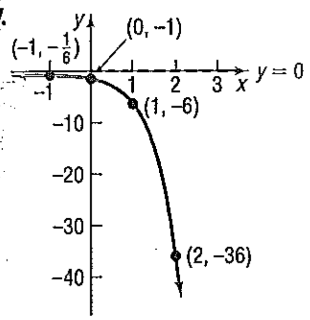

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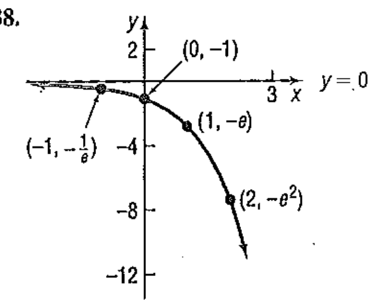


89. Find an exponential function with horizontal asymptote $y = 2$ whose graph contains the points $(0, 3)$ and $(1, 5)$.

90. Find an exponential function with horizontal asymptote $y = -3$ whose graph contains the points $(0, -2)$ and $(-2, 1)$.

Mixed Practice

91. Suppose that $f(x) = 2^x$.
 (a) What is $f(4)$? What point is on the graph of f?
 (b) If $f(x) = \frac{1}{16}$, what is x? What point is on the graph of f?

92. Suppose that $f(x) = 3^x$.
 (a) What is $f(4)$? What point is on the graph of f?
 (b) If $f(x) = \frac{1}{9}$, what is x? What point is on the graph of f?

93. Suppose that $g(x) = 4^x + 2$.
 (a) What is $g(-1)$? What point is on the graph of g?
 (b) If $g(x) = 66$, what is x? What point is on the graph of g?

94. Suppose that $g(x) = 5^x - 3$.
 (a) What is $g(-1)$? What point is on the graph of g?
 (b) If $g(x) = 122$, what is x? What point is on the graph of g?

95. Suppose that $H(x) = \left(\frac{1}{2}\right)^x - 4$.
 (a) What is $H(-6)$? What point is on the graph of H?
 (b) If $H(x) = 12$, what is x? What point is on the graph of H?
 (c) Find the zero of H.

96. Suppose that $F(x) = \left(\frac{1}{3}\right)^x - 3$.
 (a) What is $F(-5)$? What point is on the graph of F?
 (b) If $F(x) = 24$, what is x? What point is on the graph of F?
 (c) Find the zero of F.

In Problems 97–100, graph each function. Based on the graph, state the domain and the range, and find any intercepts.

97. $f(x) = \begin{cases} e^{-x} & \text{if } x < 0 \\ e^{x} & \text{if } x \geq 0 \end{cases}$

98. $f(x) = \begin{cases} e^{x} & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$

99. $f(x) = \begin{cases} -e^{x} & \text{if } x < 0 \\ -e^{-x} & \text{if } x \geq 0 \end{cases}$

100. $f(x) = \begin{cases} -e^{-x} & \text{if } x < 0 \\ -e^{x} & \text{if } x \geq 0 \end{cases}$

Applications and Extensions

101. **Optics** If a single pane of glass obliterates 3% of the light passing through it, the percent p of light that passes through n successive panes is given approximately by the function

$$p(n) = 100(0.97)^n$$

(a) What percent of light will pass through 10 panes?
(b) What percent of light will pass through 25 panes?
(c) Explain the meaning of the base 0.97 in this problem.

102. **Atmospheric Pressure** The atmospheric pressure p on a balloon or plane decreases with increasing height. This pressure, measured in millimeters of mercury, is related to the height h (in kilometers) above sea level by the function

$$p(h) = 760e^{-0.145h}$$

(a) Find the atmospheric pressure at a height of 2 kilometers (over a mile).
(b) What is it at a height of 10 kilometers (over 30,000 feet)?

103. **Depreciation** The price p, in dollars, of a Honda Civic EX-L Sedan that is x years old is modeled by

$$p(x) = 22{,}265(0.90)^x$$

(a) How much should a 3-year-old Civic EX-L Sedan cost?
(b) How much should a 9-year-old Civic EX-L Sedan cost?
(c) Explain the meaning of the base 0.90 in this problem.

104. **Healing of Wounds** The normal healing of wounds can be modeled by an exponential function. If A_0 represents the original area of the wound and if A equals the area of the wound, then the function

$$A(n) = A_0e^{-0.35n}$$

describes the area of a wound after n days following an injury when no infection is present to retard the healing. Suppose that a wound initially had an area of 100 square millimeters.
(a) If healing is taking place, how large will the area of the wound be after 3 days?
(b) How large will it be after 10 days?

105. **Advanced-Stage Pancreatic Cancer** The percentage of patients P who have survived t years after initial diagnosis of advanced-stage pancreatic cancer is modeled by the function

$$P(t) = 100(0.3)^t$$

Source: Cancer Treatment Centers of America
(a) According to the model, what percent of patients survive 1 year after initial diagnosis?
(b) What percent of patients survive 2 years after initial diagnosis?
(c) Explain the meaning of the base, 0.3, in the context of this problem.

106. **Endangered Species** In a protected environment, the population P of an endangered species recovers over time t according to the model

$$P(t) = 30(1.149)^t$$

(a) What is the size of the initial population of the species?
(b) According to the model, what will be the population of the species in 5 years?
(c) According to the model, what will be the population of the species in 10 years?
(d) According to the model, what will be the population of the species in 15 years?
(e) What is happening to the population every 5 years?

107. **Drug Medication** The function

$$D(h) = 5e^{-0.4h}$$

can be used to find the number of milligrams D of a certain drug that is in a patient's bloodstream h hours after the drug has been administered. How many milligrams will be present after 1 hour? After 6 hours?

108. **Spreading of Rumors** A model for the number N of people in a college community who have heard a certain rumor is

$$N = P(1 - e^{-0.15d})$$

where P is the total population of the community and d is the number of days that have elapsed since the rumor began. In a community of 1000 students, how many students will have heard the rumor after 3 days?

109. **Exponential Probability** Between 12:00 PM and 1:00 PM, cars arrive at Citibank's drive-thru at the rate of 6 cars per hour (0.1 car per minute). The following formula from probability can be used to determine the probability that a car will arrive within t minutes of 12:00 PM:

$$F(t) = 1 - e^{-0.1t}$$

(a) Determine the probability that a car will arrive within 10 minutes of 12:00 PM (that is, before 12:10 PM).
(b) Determine the probability that a car will arrive within 40 minutes of 12:00 PM (before 12:40 PM).
(c) What value does F approach as t becomes unbounded in the positive direction?
(d) Graph F using a graphing utility.
(e) Using INTERSECT, determine how many minutes are needed for the probability to reach 50%.

110. **Exponential Probability** Between 5:00 PM and 6:00 PM, cars arrive at Jiffy Lube at the rate of 9 cars per hour (0.15 car per minute). The following formula from probability can be used to determine the probability that a car will arrive within t minutes of 5:00 PM:

$$F(t) = 1 - e^{-0.15t}$$

(a) Determine the probability that a car will arrive within 15 minutes of 5:00 PM (that is, before 5:15 PM).
(b) Determine the probability that a car will arrive within 30 minutes of 5:00 PM (before 5:30 PM).
(c) What value does F approach as t becomes unbounded in the positive direction?

(d) Graph F using a graphing utility.
(e) Using INTERSECT, determine how many minutes are needed for the probability to reach 60%.

111. Poisson Probability Between 5:00 PM and 6:00 PM, cars arrive at McDonald's drive-thru at the rate of 20 cars per hour. The following formula from probability can be used to determine the probability that x cars will arrive between 5:00 PM and 6:00 PM.

$$P(x) = \frac{20^x e^{-20}}{x!}$$

where

$$x! = x \cdot (x-1) \cdot (x-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

(a) Determine the probability that $x = 15$ cars will arrive between 5:00 PM and 6:00 PM.
(b) Determine the probability that $x = 20$ cars will arrive between 5:00 PM and 6:00 PM.

112. Poisson Probability People enter a line for the *Demon Roller Coaster* at the rate of 4 per minute. The following formula from probability can be used to determine the probability that x people will arrive within the next minute.

$$P(x) = \frac{4^x e^{-4}}{x!}$$

where

$$x! = x \cdot (x-1) \cdot (x-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

(a) Determine the probability that $x = 5$ people will arrive within the next minute.
(b) Determine the probability that $x = 8$ people will arrive within the next minute.

113. Relative Humidity The relative humidity is the ratio (expressed as a percent) of the amount of water vapor in the air to the maximum amount that it can hold at a specific temperature. The relative humidity, R, is found using the following formula:

$$R = 10^{\frac{4221}{T+459.4} - \frac{4221}{D+459.4} + 2}$$

where T is the air temperature (in °F) and D is the dew point temperature (in °F).
(a) Determine the relative humidity if the air temperature is 50° Fahrenheit and the dew point temperature is 41° Fahrenheit.
(b) Determine the relative humidity if the air temperature is 68° Fahrenheit and the dew point temperature is 59° Fahrenheit.
(c) What is the relative humidity if the air temperature and the dew point temperature are the same?

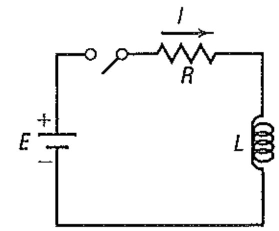
114. Learning Curve Suppose that a student has 500 vocabulary words to learn. If the student learns 15 words after 5 minutes, the function

$$L(t) = 500(1 - e^{-0.0061t})$$

approximates the number of words L that the student will have learned after t minutes.
(a) How many words will the student have learned after 30 minutes?
(b) How many words will the student have learned after 60 minutes?

115. Current in a *RL* Circuit The equation governing the amount of current I (in amperes) after time t (in seconds) in a single RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is

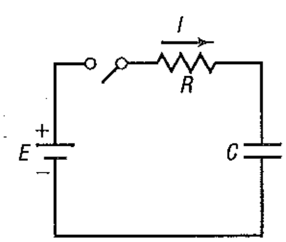
$$I = \frac{E}{R}[1 - e^{-(R/L)t}]$$



(a) If $E = 120$ volts, $R = 10$ ohms, and $L = 5$ henrys, how much current I_1 is flowing after 0.3 second? After 0.5 second? After 1 second?
(b) What is the maximum current?
(c) Graph this function $I = I_1(t)$, measuring I along the y-axis and t along the x-axis.
(d) If $E = 120$ volts, $R = 5$ ohms, and $L = 10$ henrys, how much current I_2 is flowing after 0.3 second? After 0.5 second? After 1 second?
(e) What is the maximum current?
(f) Graph the function $I = I_2(t)$ on the same coordinate axes as $I_1(t)$.

116. Current in a *RC* Circuit The equation governing the amount of current I (in amperes) after time t (in microseconds) in a single RC circuit consisting of a resistance R (in ohms), a capacitance C (in microfarads), and an electromotive force E (in volts) is

$$I = \frac{E}{R} e^{-t/(RC)}$$



(a) If $E = 120$ volts, $R = 2000$ ohms, and $C = 1.0$ microfarad, how much current I_1 is flowing initially ($t = 0$)? After 1000 microseconds? After 3000 microseconds?
(b) What is the maximum current?
(c) Graph the function $I = I_1(t)$, measuring I along the y-axis and t along the x-axis.
(d) If $E = 120$ volts, $R = 1000$ ohms, and $C = 2.0$ microfarads, how much current I_2 is flowing initially? After 1000 microseconds? After 3000 microseconds?
(e) What is the maximum current?
(f) Graph the function $I = I_2(t)$ on the same coordinate axes as $I_1(t)$.

117. If f is an exponential function of the form $f(x) = C \cdot a^x$ with growth factor 3 and $f(6) = 12$, what is $f(7)$?

118. Another Formula for e Use a calculator to compute the values of

$$2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

for $n = 4, 6, 8$, and 10. Compare each result with e.
[**Hint:** $1! = 1, 2! = 2 \cdot 1, 3! = 3 \cdot 2 \cdot 1$,

$$n! = n(n-1) \cdot \cdots \cdot (3)(2)(1).]$$

119. Another Formula for e Use a calculator to compute the various values of the expression. Compare the values to e.

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{\text{etc.}}}}}}$$

120. Difference Quotient If $f(x) = a^x$, show that

$$\frac{f(x+h) - f(x)}{h} = a^x \cdot \frac{a^h - 1}{h} \quad h \neq 0$$

121. If $f(x) = a^x$, show that $f(A + B) = f(A) \cdot f(B)$.

122. If $f(x) = a^x$, show that $f(-x) = \dfrac{1}{f(x)}$.

123. If $f(x) = a^x$, show that $f(\alpha x) = [f(x)]^\alpha$.

Problems 124 and 125 provide definitions for two other transcendental functions.

124. The **hyperbolic sine function,** designated by $\sinh x$, is defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

(a) Show that $f(x) = \sinh x$ is an odd function.
(b) Graph $f(x) = \sinh x$ using a graphing utility.

125. The **hyperbolic cosine function,** designated by $\cosh x$, is defined as

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

(a) Show that $f(x) = \cosh x$ is an even function.
(b) Graph $f(x) = \cosh x$ using a graphing utility.
(c) Refer to Problem 124. Show that, for every x,

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

126. Historical Problem Pierre de Fermat (1601–1665) conjectured that the function

$$f(x) = 2^{(2^x)} + 1$$

for $x = 1, 2, 3, \ldots$, would always have a value equal to a prime number. But Leonhard Euler (1707–1783) showed that this formula fails for $x = 5$. Use a calculator to determine the prime numbers produced by f for $x = 1, 2, 3, 4$. Then show that $f(5) = 641 \times 6{,}700{,}417$, which is not prime.

Discussion and Writing

127. The bacteria in a 4-liter container double every minute. After 60 minutes the container is full. How long did it take to fill half the container?

128. Explain in your own words what the number e is. Provide at least two applications that use this number.

129. Do you think that there is a power function that increases more rapidly than an exponential function whose base is greater than 1? Explain.

130. As the base a of an exponential function $f(x) = a^x$, where $a > 1$, increases, what happens to the behavior of its graph for $x > 0$? What happens to the behavior of its graph for $x < 0$?

131. The graphs of $y = a^{-x}$ and $y = \left(\dfrac{1}{a}\right)^x$ are identical. Why?

Retain Your Knowledge

Problems 132–135 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

132. Solve the inequality: $x^3 + 5x^2 \le 4x + 20$

133. Solve the inequality: $\dfrac{x+1}{x-2} \ge 1$

134. Determine whether the function is linear or nonlinear. If linear, determine the equation that defines $y = f(x)$.

x	$y = f(x)$
−6	−3
−3	−4
0	−5
3	−6
6	−7

135. Consider the quadratic function $f(x) = x^2 + 2x - 3$.
(a) Graph f by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.
(b) Determine the domain and the range of f.
(c) Determine where f is increasing and where it is decreasing.

'Are You Prepared?' Answers

1. $64; 4; \frac{1}{9}$ **2.** $\{-4, 1\}$ **3.** False **4.** 3 **5.** True