Historical Feature

John Napier (1550–1617)

Logarithms were invented about 1590 by John Napier (1550–1617) and Joost Bürgi (1552–1632), working independently. Napier, whose work had the greater influence, was a Scottish lord, a secretive man whose neighbors were inclined to believe him to be in league with the devil. His approach to logarithms was very different from ours; it was based on the relationship between arithmetic and geometric sequences, discussed in a later chapter, and not on the inverse function relationship of logarithms to exponential functions (described in Section 4.4). Napier's tables, published in 1614, listed what would now be called *natural logarithms* of sines and were rather difficult to use. A London professor, Henry Briggs, became interested in the tables and visited Napier. In their conversations, they developed the idea of common logarithms, which were published in 1617. Their importance for calculation was immediately recognized, and by 1650 they were being printed as far away as China. They remained an important calculation tool until the advent of the inexpensive handheld calculator about 1972, which has decreased their calculational, but not their theoretical, importance.

A side effect of the invention of logarithms was the popularization of the decimal system of notation for real numbers.

4.5 Assess Your Understanding

Concepts and Vocabulary

1. $\log_a 1 =$ ____ **2.** $\log_a a =$ ____ **3.** $a^{\log_a M} =$ ____ **4.** $\log_a a^r =$ ____

5. $\log_a(MN) =$ ________ + ________ **6.** $\log_a\left(\frac{M}{N}\right) =$ ________ − ________

7. $\log_a M^r =$ ________ **8.** If $\log_a x = \log_a 6$, then $x =$ ____.

9. If $\log_8 M = \frac{\log_5 7}{\log_5 8}$, then $M =$ ____. **10.** ***True or False*** $\ln(x+3) - \ln(2x) = \frac{\ln(x+3)}{\ln(2x)}$

11. ***True or False*** $\log_2(3x^4) = 4\log_2(3x)$ **12.** ***True or False*** $\frac{\ln 8}{\ln 4} = 2$

Skill Building

In Problems 13–28, use properties of logarithms to find the exact value of each expression. Do not use a calculator.

13. $\log_3 3^{71}$ **14.** $\log_2 2^{-13}$ **15.** $\ln e^{-4}$ **16.** $\ln e^{\sqrt{2}}$

17. $2^{\log_2 7}$ **18.** $e^{\ln 8}$ **19.** $\log_8 2 + \log_8 4$ **20.** $\log_6 9 + \log_6 4$

21. $\log_6 18 - \log_6 3$ **22.** $\log_8 16 - \log_8 2$ **23.** $\log_2 6 \cdot \log_6 8$ **24.** $\log_3 8 \cdot \log_8 9$

25. $3^{\log_3 5 - \log_3 4}$ **26.** $5^{\log_5 6 + \log_5 7}$ **27.** $e^{\log_{e^2} 16}$ **28.** $e^{\log_{e^2} 9}$

In Problems 29–36, suppose that $\ln 2 = a$ *and* $\ln 3 = b$. *Use properties of logarithms to write each logarithm in terms of a and b.*

29. $\ln 6$ **30.** $\ln\frac{2}{3}$ **31.** $\ln 1.5$ **32.** $\ln 0.5$

33. $\ln 8$ **34.** $\ln 27$ **35.** $\ln\sqrt[5]{6}$ **36.** $\ln\sqrt[4]{\frac{2}{3}}$

In Problems 37–56, write each expression as a sum and/or difference of logarithms. Express powers as factors.

37. $\log_5(25x)$ **38.** $\log_3\frac{x}{9}$ **39.** $\log_2 z^3$ **40.** $\log_7 x^5$

41. $\ln(ex)$ **42.** $\ln\frac{e}{x}$ **43.** $\ln\frac{x}{e^x}$ **44.** $\ln(xe^x)$

45. $\log_a(u^2v^3)$ $u > 0, v > 0$ **46.** $\log_2\left(\frac{a}{b^2}\right)$ $a > 0, b > 0$ **47.** $\ln(x^2\sqrt{1-x})$ $0 < x < 1$ **48.** $\ln(x\sqrt{1+x^2})$ $x > 0$

49. $\log_2\left(\frac{x^3}{x-3}\right)$ $x > 3$ **50.** $\log_5\left(\frac{\sqrt[3]{x^2+1}}{x^2-1}\right)$ $x > 1$ **51.** $\log\left[\frac{x(x+2)}{(x+3)^2}\right]$ $x > 0$ **52.** $\log\left[\frac{x^3\sqrt{x+1}}{(x-2)^2}\right]$ $x > 2$

53. $\ln\left[\frac{x^2-x-2}{(x+4)^2}\right]^{1/3}$ $x > 2$ **54.** $\ln\left[\frac{(x-4)^2}{x^2-1}\right]^{2/3}$ $x > 4$ **55.** $\ln\frac{5x\sqrt{1+3x}}{(x-4)^3}$ $x > 4$ **56.** $\ln\left[\frac{5x^2\sqrt[3]{1-x}}{4(x+1)^2}\right]$ $0 < x < 1$

In Problems 57–70, write each expression as a single logarithm.

57. $3\log_5 u + 4\log_5 v$ **58.** $2\log_3 u - \log_3 v$ **59.** $\log_3\sqrt{x} - \log_3 x^3$

60. $\log_2\left(\frac{1}{x}\right) + \log_2\left(\frac{1}{x^2}\right)$ **61.** $\log_4(x^2-1) - 5\log_4(x+1)$ **62.** $\log(x^2+3x+2) - 2\log(x+1)$

63. $\ln\left(\frac{x}{x-1}\right) + \ln\left(\frac{x+1}{x}\right) - \ln(x^2-1)$ **64.** $\log\left(\frac{x^2+2x-3}{x^2-4}\right) - \log\left(\frac{x^2+7x+6}{x+2}\right)$ **65.** $8\log_2\sqrt{3x-2} - \log_2\left(\frac{4}{x}\right) + \log_2 4$

66. $21 \log_3 \sqrt[3]{x} + \log_3(9x^2) - \log_3 9$ **67.** $2 \log_a(5x^3) - \frac{1}{2}\log_a(2x + 3)$ **68.** $\frac{1}{3}\log(x^3 + 1) + \frac{1}{2}\log(x^2 + 1)$

69. $2 \log_2(x + 1) - \log_2(x + 3) - \log_2(x - 1)$ **70.** $3 \log_5(3x + 1) - 2 \log_5(2x - 1) - \log_5 x$

In Problems 71–78, use the Change-of-Base Formula and a calculator to evaluate each logarithm. Round your answer to three decimal places.

71. $\log_3 21$ **72.** $\log_5 18$ **73.** $\log_{1/3} 71$ **74.** $\log_{1/2} 15$

75. $\log_{\sqrt{2}} 7$ **76.** $\log_{\sqrt{5}} 8$ **77.** $\log_\pi e$ **78.** $\log_\pi \sqrt{2}$

In Problems 79–84, graph each function using a graphing utility and the Change-of-Base Formula.

79. $y = \log_4 x$ **80.** $y = \log_5 x$ **81.** $y = \log_2(x + 2)$ **82.** $y = \log_4(x - 3)$ **83.** $y = \log_{x-1}(x + 1)$ **84.** $y = \log_{x+2}(x - 2)$

Mixed Practice

85. If $f(x) = \ln x, g(x) = e^x$, and $h(x) = x^2$, find:
(a) $(f \circ g)(x)$. What is the domain of $f \circ g$?
(b) $(g \circ f)(x)$. What is the domain of $g \circ f$?
(c) $(f \circ g)(5)$
(d) $(f \circ h)(x)$. What is the domain of $f \circ h$?
(e) $(f \circ h)(e)$

86. If $f(x) = \log_2 x, g(x) = 2^x$, and $h(x) = 4x$, find:
(a) $(f \circ g)(x)$. What is the domain of $f \circ g$?
(b) $(g \circ f)(x)$. What is the domain of $g \circ f$?
(c) $(f \circ g)(3)$
(d) $(f \circ h)(x)$. What is the domain of $f \circ h$?
(e) $(f \circ h)(8)$

Applications and Extensions

In Problems 87–96, express y as a function of x. The constant C is a positive number.

87. $\ln y = \ln x + \ln C$

88. $\ln y = \ln(x + C)$

89. $\ln y = \ln x + \ln(x + 1) + \ln C$

90. $\ln y = 2 \ln x - \ln(x + 1) + \ln C$

91. $\ln y = 3x + \ln C$

92. $\ln y = -2x + \ln C$

93. $\ln(y - 3) = -4x + \ln C$

94. $\ln(y + 4) = 5x + \ln C$

95. $3 \ln y = \frac{1}{2}\ln(2x + 1) - \frac{1}{3}\ln(x + 4) + \ln C$

96. $2 \ln y = -\frac{1}{2}\ln x + \frac{1}{3}\ln(x^2 + 1) + \ln C$

97. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$.

98. Find the value of $\log_2 4 \cdot \log_4 6 \cdot \log_6 8$.

99. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \cdots \cdot \log_n(n + 1) \cdot \log_{n+1} 2$.

100. Find the value of $\log_2 2 \cdot \log_2 4 \cdot \cdots \cdot \log_2 2^n$.

101. Show that $\log_a(x + \sqrt{x^2 - 1}) + \log_a(x - \sqrt{x^2 - 1}) = 0$.

102. Show that $\log_a(\sqrt{x} + \sqrt{x - 1}) + \log_a(\sqrt{x} - \sqrt{x - 1}) = 0$.

103. Show that $\ln(1 + e^{2x}) = 2x + \ln(1 + e^{-2x})$.

104. Difference Quotient If $f(x) = \log_a x$, show that $\dfrac{f(x + h) - f(x)}{h} = \log_a\left(1 + \dfrac{h}{x}\right)^{1/h}, \quad h \neq 0.$

105. If $f(x) = \log_a x$, show that $-f(x) = \log_{1/a} x$.

106. If $f(x) = \log_a x$, show that $f(AB) = f(A) + f(B)$.

107. If $f(x) = \log_a x$, show that $f\left(\frac{1}{x}\right) = -f(x)$.

108. If $f(x) = \log_a x$, show that $f(x^\alpha) = \alpha f(x)$.

109. Show that $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$, where a, M, and N are positive real numbers and $a \neq 1$.

110. Show that $\log_a\left(\frac{1}{N}\right) = -\log_a N$, where a and N are positive real numbers and $a \neq 1$.

Discussion and Writing

111. Graph $Y_1 = \log(x^2)$ and $Y_2 = 2\log(x)$ using a graphing utility. Are they equivalent? What might account for any differences in the two functions?

112. Write an example that illustrates why $(\log_a x)^r \neq r \log_a x$.

113. Write an example that illustrates why $\log_2(x + y) \neq \log_2 x + \log_2 y$.

114. Does $3^{\log_3(-5)} = -5$? Why or why not?

Retain Your Knowledge

Problems 115–118 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

115. Use a graphing utility to solve $x^3 - 3x^2 - 4x + 8 = 0$. Round answers to two decimal places.

116. Find the real zeros of $f(x) = 5x^5 + 44x^4 + 116x^3 + 95x^2 - 4x - 4$

117. Without solving, determine the character of the solution of the quadratic equation $4x^2 - 28x + 49 = 0$ in the complex system.

118. Graph $f(x) = \sqrt{2 - x}$ using the techniques of shifting, compressing or stretching, and reflections. State the domain and the range of f.