PreCalc KEY Dec 7, 2016
NOTES

4.8 Exponential Growth & Decay Models & Newton's Law of Cooling

I. FIND EQUATIONS OF POPULATIONS THAT OBEY THE LAW OF UNINHIBITED GROWTH

$$A(t) = A_o e^{kt}$$

Where A_o = original amount, k = constant (growth or decay rate), where $k \neq 0$, $k > 0$ implies growth, and $k < 0$ implies decay.

Uninhibited growth early stages of cell division, growth of organisms, and plants, etc. under the assumption that no cells die.

Example 1. A colony of bacteria grows according to the law of uninhibited growth according to the function $N(t) = 100e^{0.045t}$, where N is measured in grams and t is measured in days.

a) Determine the initial amount of bacteria.
b) What is the growth rate of the bacteria?
c) What is the population after 5 days?
d) How long will it take for the population to reach 140 grams?
e) What is the doubling time for the population?

Solution:

a) 100, because $A_0 = 100$ or @ $t=0$; $N(0) = 100e^{.045(0)} = 100(1) = 100$.

b) $.045 \cdot 100 = 4.5\%$ growth rate

c) $N(5) = 100e^{0.045(5)}$

$\approx 125.2g$

d) $140 = 100e^{0.045t}$

$1.4 = e^{0.045t}$

$\ln(1.4) = \ln(e^{.045t})$

$\frac{\ln(1.4)}{.045} = \frac{.045t}{.045}$

$t \approx 7.5$ days

e) $2\cancel{N} = \cancel{N}e^{.045t}$

$2 = e^{.045t}$

OR

$200 = 100e^{.045t}$

$2 = e^{.045t}$

$\frac{\ln(2)}{.045} = \frac{\cancel{.045}t}{\cancel{.045}}$

$t \approx 15.4$ days

Example 2. A colony of bacteria increases according to the law of uninhibited growth.

a) If N is the number of cells and t is the time in hours, express N as a function of t.
b) If the number of bacteria doubles in 3 hours, find the function that gives the number of cells in the culture.
c) How long will it take for the size of the colony to triple?
d) How long will it take for the population to double a second time (that is, increase four times)?

Solution:

a) $N(t) = N_0 e^{kt}$

b) $2\cancel{N_0} = \cancel{N_0} e^{k3}$

$2 = e^{3k}$

$\ln(2) = \cancel{\ln}(\cancel{e}^{k3})$

$\ln(2) = 3k$

$\frac{\ln(2)}{3} = k$

$k \approx .231049$

$\boxed{N(t) = N_0 e^{.231049t}}$

c) $3 = e^{.231049t}$

$[3\cancel{N_0} = \cancel{N_0} e^{.231049t}]$

$\ln(3) = \cancel{\ln}(\cancel{e}^{.231049t})$

$\frac{\ln(3)}{.231049} = \frac{\cancel{.231049}t}{\cancel{.231049}}$

$\boxed{t \approx 4.75 \text{ hrs}}$

d) $4\cancel{N_0} = \cancel{N_0} e^{.231049t}$

$4 = e^{.231049t}$

$\ln(4) = \cancel{\ln}(\cancel{e}^{.231049t})$

$\frac{\ln(4)}{.231049} = \frac{\cancel{.231049}t}{\cancel{.231049}}$

$\boxed{t \approx 6 \text{ hrs}}$

II. FIND EQUATIONS OF POPULATIONS THAT OBEY THE LAW OF DECAY

The amount A of a radioactive material present at time t is given by

$$A(t) = A_o e^{kt} \qquad k < 0$$

Half-life is the time required for half of the radioactive substance to decay.

Carbon dating the ratio of carbon 12 and carbon 14 is constant for live organisms, but after death the amount of carbon 12 remains constant and carbon 14 decreases.

Example 3. Traces of burned wood along with ancient stone tools in an archeological dig in Chile were found to contain approximately 1.6% of the original amount of carbon 14. If the half-life of carbon 14 is 5730 years, approximately when was the tree cut and burned?

first, we need to find K

$$\tfrac{1}{2}A_0 = A_0 e^{k(5730)}$$

$$\tfrac{1}{2} = e^{k(5730)}$$

$$\ln\left(\tfrac{1}{2}\right) = 5730k$$

$$k = \frac{\ln(.5)}{5730} \approx -.000120968$$

So, $A = A_0 e^{-.000120968t}$ and $1.67\% = .0167$

$$.0167A_0 = A_0 e^{-.000120968t}$$

$$.0167 = e^{-.000120968t}$$

$$\ln(.0167) = -.000120968t$$

$$t = \frac{\ln(.0167)}{-.000120968} \approx \boxed{33{,}830 \text{ yrs}}$$

III. NEWTON'S LAW OF COOLING

....states that the temperature of a heated object decreases exponentially over time toward the temperature of the surrounding medium (i.e. the ambient temperature).

The temperature u of a heated object at a given time t can be modeled by the following function:

$$u(t) = T + (u_o - T)e^{kt} \qquad k < 0$$

Where T is the constant temperature of the surrounding medium, u_o is the initial temperature of the heated object, and k is a negative constant.

Example 4. An object is heated to 100°C and is then allowed to cool in a room whose air temperature is 30°C.

a) If the temperature of the object is 80°C after 5 minutes, when will its temperature be 50°C?
b) Determine the elapsed time before the temperature of the object is 35°C?
c) What do you notice about the temperature as time passes?

Solution:

note: $u = 80°$ @ $t = 5$, need k, $T = 30°$, $u_0 = 100°$

a) $u(t) = 30 + (100 - 30)e^{kt}$

$= 30 + 70e^{kt}$

$80 = 30 + 70e^{k(5)}$

$50 = 70e^{5k}$

$e^{5k} = 50/70$

$\ln(e^{5k}) = \ln(5/7)$

$5k = \ln(5/7)$

$k = \dfrac{\ln(5/7)}{5}$

$\approx -.0673$

$\therefore u(t) = 30 + 70e^{-.0673t}$

$50 = 30 + 70e^{-.0673t}$

$20 = 70e^{-.0673t}$

$2/7 = e^{-.0673t}$

$\ln(2/7) = -.0673t$

$t = \dfrac{\ln(2/7)}{-.0673} \approx 18.6$ minutes

b) $35 = 30 + 70e^{-.0673t}$

$5 = 70e^{-.0673t}$

$5/70 = e^{-.0673t}$

$\ln(5/70) = -.0673t$

$t = \dfrac{\ln(5/70)}{-.0673} \approx 39.2$ min

c) as time passes, the temperature approaches the room temp, 30°C.