Now, using these results and some of the fundamental identities, we have

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta \qquad \cot(-\theta) = \frac{1}{\tan(-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta$$

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta \qquad \csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta$$

■

EXAMPLE 7 **Finding Exact Values Using Even–Odd Properties**

Find the exact value of:

(a) $\sin(-45°)$ (b) $\cos(-\pi)$ (c) $\cot\left(-\frac{3\pi}{2}\right)$ (d) $\tan\left(-\frac{37\pi}{4}\right)$

Solution

(a) $\sin(-45°) = -\sin 45° = -\frac{\sqrt{2}}{2}$ (Odd function)

(b) $\cos(-\pi) = \cos\pi = -1$ (Even function)

(c) $\cot\left(-\frac{3\pi}{2}\right) = -\cot\frac{3\pi}{2} = 0$ (Odd function)

(d) $\tan\left(-\frac{37\pi}{4}\right) = -\tan\frac{37\pi}{4} = -\tan\left(\frac{\pi}{4} + 9\pi\right) = -\tan\frac{\pi}{4} = -1$ (Odd function; Period is π.)

●

Now Work PROBLEM 59

5.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The domain of the function $f(x) = \frac{x+1}{2x+1}$ is ________. (pp. 49–51)
2. A function for which $f(x) = f(-x)$ for all x in the domain of f is called a(n) _____ function. (pp. 66–68)
3. ***True or False*** The function $f(x) = \sqrt{x}$ is even. (pp. 66–68)
4. ***True or False*** The equation $x^2 + 2x = (x+1)^2 - 1$ is an identity. (p. A63)

Concepts and Vocabulary

5. The sine, cosine, cosecant, and secant functions have period _______; the tangent and cotangent functions have period _______.
6. The domain of the tangent function is _______.
7. The range of the sine function is ___________.
8. ***True or False*** The only even trigonometric functions are the cosine and secant functions.
9. $\sin^2\theta + \cos^2\theta =$ ___
10. ***True or False*** $\sec\theta = \frac{1}{\sin\theta}$

Skill Building

In Problems 11–26, use the fact that the trigonometric functions are periodic to find the exact value of each expression. Do not use a calculator.

11. $\sin 405°$
12. $\cos 420°$
13. $\tan 405°$
14. $\sin 390°$
15. $\csc 450°$
16. $\sec 540°$
17. $\cot 390°$
18. $\sec 420°$
19. $\cos\frac{33\pi}{4}$
20. $\sin\frac{9\pi}{4}$
21. $\tan(21\pi)$
22. $\csc\frac{9\pi}{2}$
23. $\sec\frac{17\pi}{4}$
24. $\cot\frac{17\pi}{4}$
25. $\tan\frac{19\pi}{6}$
26. $\sec\frac{25\pi}{6}$

In Problems 27–34, name the quadrant in which the angle θ lies.

27. $\sin\theta > 0$, $\cos\theta < 0$
28. $\sin\theta < 0$, $\cos\theta > 0$
29. $\sin\theta < 0$, $\tan\theta < 0$
30. $\cos\theta > 0$, $\tan\theta > 0$
31. $\cos\theta > 0$, $\tan\theta < 0$
32. $\cos\theta < 0$, $\tan\theta > 0$
33. $\sec\theta < 0$, $\sin\theta > 0$
34. $\csc\theta > 0$, $\cos\theta < 0$

In Problems 35–42, $\sin\theta$ and $\cos\theta$ are given. Find the exact value of each of the four remaining trigonometric functions.

35. $\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$
36. $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$
37. $\sin\theta = \frac{2\sqrt{5}}{5}$, $\cos\theta = \frac{\sqrt{5}}{5}$
38. $\sin\theta = -\frac{\sqrt{5}}{5}$, $\cos\theta = -\frac{2\sqrt{5}}{5}$
39. $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$
40. $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$
41. $\sin\theta = -\frac{1}{3}$, $\cos\theta = \frac{2\sqrt{2}}{3}$
42. $\sin\theta = \frac{2\sqrt{2}}{3}$, $\cos\theta = -\frac{1}{3}$

In Problems 43–58, find the exact value of each of the remaining trigonometric functions of θ.

43. $\sin\theta = \frac{12}{13}$, θ in quadrant II
44. $\cos\theta = \frac{3}{5}$, θ in quadrant IV
45. $\cos\theta = -\frac{4}{5}$, θ in quadrant III
46. $\sin\theta = -\frac{5}{13}$, θ in quadrant III
47. $\sin\theta = \frac{5}{13}$, $90° < \theta < 180°$
48. $\cos\theta = \frac{4}{5}$, $270° < \theta < 360°$
49. $\cos\theta = -\frac{1}{3}$, $\frac{\pi}{2} < \theta < \pi$
50. $\sin\theta = -\frac{2}{3}$, $\pi < \theta < \frac{3\pi}{2}$
51. $\sin\theta = \frac{2}{3}$, $\tan\theta < 0$
52. $\cos\theta = -\frac{1}{4}$, $\tan\theta > 0$
53. $\sec\theta = 2$, $\sin\theta < 0$
54. $\csc\theta = 3$, $\cot\theta < 0$
55. $\tan\theta = \frac{3}{4}$, $\sin\theta < 0$
56. $\cot\theta = \frac{4}{3}$, $\cos\theta < 0$
57. $\tan\theta = -\frac{1}{3}$, $\sin\theta > 0$
58. $\sec\theta = -2$, $\tan\theta > 0$

In Problems 59–76, use the even–odd properties to find the exact value of each expression. Do not use a calculator.

59. $\sin(-60°)$
60. $\cos(-30°)$
61. $\tan(-30°)$
62. $\sin(-135°)$
63. $\sec(-60°)$
64. $\csc(-30°)$
65. $\sin(-90°)$
66. $\cos(-270°)$
67. $\tan\left(-\frac{\pi}{4}\right)$
68. $\sin(-\pi)$
69. $\cos\left(-\frac{\pi}{4}\right)$
70. $\sin\left(-\frac{\pi}{3}\right)$
71. $\tan(-\pi)$
72. $\sin\left(-\frac{3\pi}{2}\right)$
73. $\csc\left(-\frac{\pi}{4}\right)$
74. $\sec(-\pi)$
75. $\sec\left(-\frac{\pi}{6}\right)$
76. $\csc\left(-\frac{\pi}{3}\right)$

In Problems 77–88, use properties of the trigonometric functions to find the exact value of each expression. Do not use a calculator.

77. $\sin^2 40° + \cos^2 40°$
78. $\sec^2 18° - \tan^2 18°$
79. $\sin 80° \csc 80°$
80. $\tan 10° \cot 10°$
81. $\tan 40° - \frac{\sin 40°}{\cos 40°}$
82. $\cot 20° - \frac{\cos 20°}{\sin 20°}$
83. $\cos 400° \cdot \sec 40°$
84. $\tan 200° \cdot \cot 20°$
85. $\sin\left(-\frac{\pi}{12}\right)\csc\frac{25\pi}{12}$
86. $\sec\left(-\frac{\pi}{18}\right)\cdot\cos\frac{37\pi}{18}$
87. $\frac{\sin(-20°)}{\cos 380°} + \tan 20°$
88. $\frac{\sin 70°}{\cos(-430°)} + \tan(-70°)$

89. If $\sin\theta = 0.3$, find the value of:
$$\sin\theta + \sin(\theta + 2\pi) + \sin(\theta + 4\pi)$$
90. If $\cos\theta = 0.2$, find the value of:
$$\cos\theta + \cos(\theta + 2\pi) + \cos(\theta + 4\pi)$$
91. If $\tan\theta = 3$, find the value of:
$$\tan\theta + \tan(\theta + \pi) + \tan(\theta + 2\pi)$$
92. If $\cot\theta = -2$, find the value of:
$$\cot\theta + \cot(\theta - \pi) + \cot(\theta - 2\pi)$$
93. Find the exact value of:
$$\sin 1° + \sin 2° + \sin 3° + \cdots + \sin 358° + \sin 359°$$
94. Find the exact value of:
$$\cos 1° + \cos 2° + \cos 3° + \cdots + \cos 358° + \cos 359°$$

95. What is the domain of the sine function?
96. What is the domain of the cosine function?
97. For what numbers θ is $f(\theta) = \tan\theta$ not defined?
98. For what numbers θ is $f(\theta) = \cot\theta$ not defined?
99. For what numbers θ is $f(\theta) = \sec\theta$ not defined?
100. For what numbers θ is $f(\theta) = \csc\theta$ not defined?
101. What is the range of the sine function?
102. What is the range of the cosine function?
103. What is the range of the tangent function?
104. What is the range of the cotangent function?
105. What is the range of the secant function?
106. What is the range of the cosecant function?
107. Is the sine function even, odd, or neither? Is its graph symmetric? If so, with respect to what?
108. Is the cosine function even, odd, or neither? Is its graph symmetric? If so, with respect to what?
109. Is the tangent function even, odd, or neither? Is its graph symmetric? If so, with respect to what?
110. Is the cotangent function even, odd, or neither? Is its graph symmetric? If so, with respect to what?
111. Is the secant function even, odd, or neither? Is its graph symmetric? If so, with respect to what?
112. Is the cosecant function even, odd, or neither? Is its graph symmetric? If so, with respect to what?

Applications and Extensions

In Problems 113–118, use the periodic and even–odd properties.

113. If $f(\theta) = \sin\theta$ and $f(a) = \frac{1}{3}$, find the exact value of:
(a) $f(-a)$ (b) $f(a) + f(a + 2\pi) + f(a + 4\pi)$

114. If $f(\theta) = \cos\theta$ and $f(a) = \frac{1}{4}$, find the exact value of:
(a) $f(-a)$ (b) $f(a) + f(a + 2\pi) + f(a - 2\pi)$

115. If $f(\theta) = \tan\theta$ and $f(a) = 2$, find the exact value of:
(a) $f(-a)$ (b) $f(a) + f(a + \pi) + f(a + 2\pi)$

116. If $f(\theta) = \cot\theta$ and $f(a) = -3$, find the exact value of:
(a) $f(-a)$ (b) $f(a) + f(a + \pi) + f(a + 4\pi)$

117. If $f(\theta) = \sec\theta$ and $f(a) = -4$, find the exact value of:
(a) $f(-a)$ (b) $f(a) + f(a + 2\pi) + f(a + 4\pi)$

118. If $f(\theta) = \csc\theta$ and $f(a) = 2$, find the exact value of:
(a) $f(-a)$ (b) $f(a) + f(a + 2\pi) + f(a + 4\pi)$

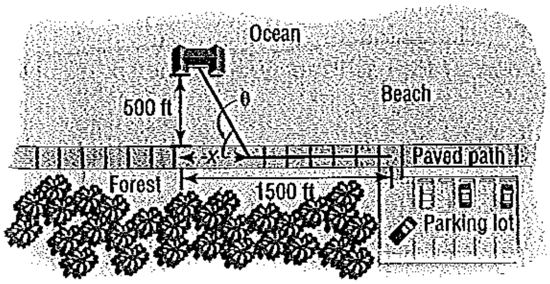
119. **Calculating the Time of a Trip** From a parking lot, you want to walk to a house on the beach. The house is located 1500 feet down a paved path that parallels the ocean, which is 500 feet away. See the illustration. Along the path you can walk 300 feet per minute, but in the sand on the beach you can only walk 100 feet per minute.

The time T to get from the parking lot to the beach house can be expressed as a function of the angle θ shown in the illustration and is

$$T(\theta) = 5 - \frac{5}{3\tan\theta} + \frac{5}{\sin\theta}, \quad 0 < \theta < \frac{\pi}{2}$$

Calculate the time T if you walk directly from the parking lot to the house.

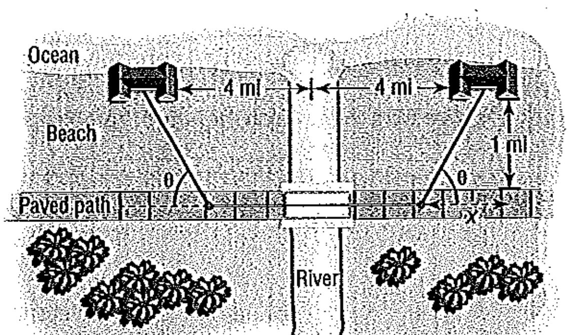
[**Hint:** $\tan\theta = \frac{500}{1500}$.]



120. **Calculating the Time of a Trip** Two oceanfront homes are located 8 miles apart on a straight stretch of beach, each a distance of 1 mile from a paved path that parallels the ocean. Sally can jog 8 miles per hour on the paved path, but only 3 miles per hour in the sand on the beach. Because a river flows directly between the two houses, it is necessary to jog in the sand to the road, continue on the path, and then jog directly back in the sand to get from one house to the other. See the illustration. The time T to get from one house to the other as a function of the angle θ shown in the illustration is

$$T(\theta) = 1 + \frac{2}{3\sin\theta} - \frac{1}{4\tan\theta} \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Calculate the time T for $\tan\theta = \frac{1}{4}$.
(b) Describe the path taken.
(c) Explain why θ must be larger than 14°.



121. Show that the range of the tangent function is the set of all real numbers.
122. Show that the range of the cotangent function is the set of all real numbers.
123. Show that the period of $f(\theta) = \sin\theta$ is 2π.
[**Hint:** Assume that $0 < p < 2\pi$ exists so that $\sin(\theta + p) = \sin\theta$ for all θ. Let $\theta = 0$ to find p. Then let $\theta = \frac{\pi}{2}$ to obtain a contradiction.]
124. Show that the period of $f(\theta) = \cos\theta$ is 2π.
125. Show that the period of $f(\theta) = \sec\theta$ is 2π.
126. Show that the period of $f(\theta) = \csc\theta$ is 2π.

127. Show that the period of $f(\theta) = \tan\theta$ is π.

128. Show that the period of $f(\theta) = \cot\theta$ is π.

129. Prove the reciprocal identities given in formula (2).

130. Prove the quotient identities given in formula (3).

131. Establish the identity:

$$(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta = 1$$

Discussion and Writing

132. Write down five properties of the tangent function. Explain the meaning of each.

133. Describe your understanding of the meaning of a periodic function.

134. Explain how to find the value of sin 390° using periodic properties.

135. Explain how to find the value of $\cos(-45°)$ using even-odd properties.

136. Explain how to find the value of sin 390° and $\cos(-45°)$ using the unit circle.

Retain Your Knowledge

Problems 137–140 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

137. Solve: $x^2 - 5 = (x+3)(x-3) + 4$

138. Graph $f(x) = -2x^2 + 12x - 13$ using transformations. Find the vertex and the axis of symmetry.

139. Solve exactly: $e^{x-4} = 6$

140. Find the real zeros of $f(x) = x^3 - 9x^2 + 3x - 27$.

'Are You Prepared?' Answers

1. $\left\{x \middle| x \neq -\frac{1}{2}\right\}$ **2.** even **3.** False **4.** True

5.4 Graphs of the Sine and Cosine Functions*

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Graphing Techniques: Transformations (Section 1.5, pp. 89–97)

Now Work the **'Are You Prepared?'** problems on page 431.

OBJECTIVES
1. Graph Functions of the Form $y = A\sin(\omega x)$ Using Transformations (p. 422)
2. Graph Functions of the Form $y = A\cos(\omega x)$ Using Transformations (p. 423)
3. Determine the Amplitude and Period of Sinusoidal Functions (p. 424)
4. Graph Sinusoidal Functions Using Key Points (p. 426)
5. Find an Equation for a Sinusoidal Graph (p. 430)

Since we want to graph the trigonometric functions in the xy-plane, we shall use the traditional symbols x for the independent variable (or argument) and y for the dependent variable (or value at x) for each function. So the six trigonometric functions can be written as

$$y = f(x) = \sin x \qquad y = f(x) = \cos x \qquad y = f(x) = \tan x$$
$$y = f(x) = \csc x \qquad y = f(x) = \sec x \qquad y = f(x) = \cot x$$

*For those who wish to include phase shifts here, Section 5.6 can be covered immediately after Section 5.4 without loss of continuity.