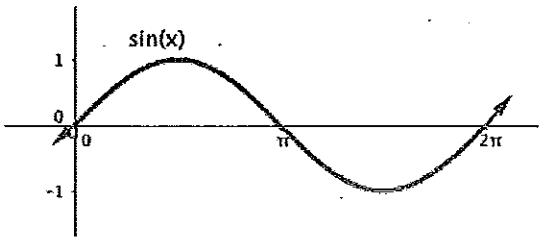
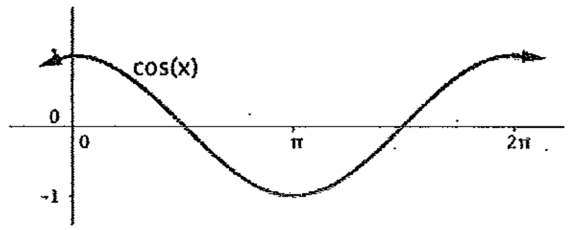
Pre Calc

5.3 Properties of the Trigonometric Functions NOTES Part I

DOMAIN





The domain of the **sine** and **cosine** function is the set of all real numbers.

The domain of the **tangent** and **secant** function is the set of all real numbers except odd integer multiples of $\pi/2$ (90°).

The domain of the **cosecant** and **cotangent** function is the set of all real numbers except integer multiples of π (180°).

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}$

RANGE

The range (span of y values) for the trig functions are as follows....

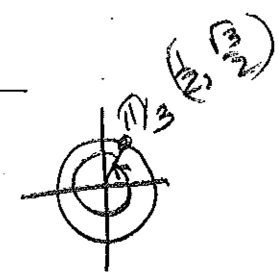
$-1 \le \sin(\theta) \le 1$ $\qquad$ $-1 \le \cos(\theta) \le 1$

$\csc(\theta) \le -1$ or $\csc(\theta) \ge 1$

$\sec(\theta) \le -1$ or $\sec(\theta) \ge 1$

$-\infty \le \tan(\theta) \le \infty$ $\qquad$ $-\infty \le \cot(\theta) \le \infty$

PERIODS OF FUNCTIONS



A "period" of a function is defined by when a full cycle is complete.

The angle of $\frac{\pi}{3}$ radians corresponds to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Notice that for angle $\frac{\pi}{3} + 2\pi$, the corresponding point on the unit circle is also $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, as the two angles are coterminal. So, we can say that $\sin(\theta) \pm 2\pi$ (and any multiple of 2π) will correspond to the same point on the unit circle of $\sin(\theta)$.

Therefore....

$$\sin(\theta + 2\pi k) = \sin(\theta) \qquad \cos(\theta + 2\pi k) = \cos(\theta)$$

where k is any integer.

The period of sine, cosine, cosecant, and secant is $\underline{2\pi}$ and the period of tangent and cotangent is $\underline{\pi}$.

Hence, the **Periodic Properties** can be stated as.....

$$\sin(\theta + 2\pi k) = \sin(\theta) \qquad \cos(\theta + 2\pi k) = \cos(\theta) \qquad \tan(\theta + \pi k) = \tan(\theta)$$

$$\csc(\theta + 2\pi k) = \csc(\theta) \qquad \sec(\theta + 2\pi k) = \sec(\theta) \qquad \cot(\theta + \pi k) = \cot(\theta)$$

where k is an integer

Example: Finding the Exact Values Using Periodic Properties.

Find the exact values of:

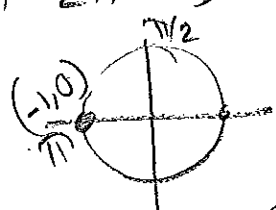
a) $\sin\frac{17\pi}{4} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\frac{17\pi}{4} - \frac{8\pi}{4} = \frac{9\pi}{4} - \frac{8\pi}{4} = \frac{\pi}{4}$

$\frac{2\pi}{1} \cdot \frac{4}{4} = \frac{8\pi}{4}$

b) $\cos(5\pi) = \cos(\pi) = -1$

$5\pi - 2\pi = 3\pi - 2\pi = \pi$



c) $\tan\frac{5\pi}{4} = \tan\left(\frac{\pi}{4}\right) = \frac{\sin \pi/4}{\cos \pi/4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$

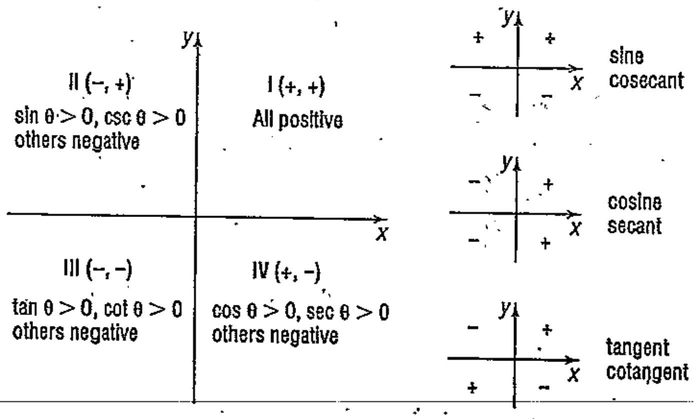
$\frac{5\pi}{4} - \frac{\pi}{1} \cdot \frac{4}{4} = \frac{5\pi}{4} - \frac{4\pi}{4} = \frac{\pi}{4}$

$\frac{\pi}{1} \cdot \frac{4}{4} = \frac{4\pi}{4}$

SIGNS OF THE TRIG FXNS IN A GIVEN QUADRANT

Knowing in which quadrant a point P lies enables us to determine the signs of the trig fxns of θ.

Quadrant of P	$\sin\theta, \csc\theta$	$\cos\theta, \sec\theta$	$\tan\theta, \cot\theta$
I	Positive	Positive	Positive
II	Positive	Negative	Negative
III	Negative	Negative	Positive
IV	Negative	Positive	Negative



(a) (b)

Example: Finding the Quadrant in Which an Angle θ Lies.

If $\sin\theta < 0$ (negative) and $\cos\theta < 0$ (negative), name the quadrant in which the angle θ lies.

III, IV   II, III

Q3

FIND THE VALUES OF THE TRIG FXNS USING FUNDAMENTAL IDENTITIES

If $P = (x, y)$ is the point on the unit circle corresponding to θ, then

$\sin(\theta) = y$ $\quad\quad$ $\csc(\theta) = \frac{1}{y}$, if $y \neq 0$

$\cos(\theta) = x$ $\quad\quad$ $\sec(\theta) = \frac{1}{x}$, if $x \neq 0$

$\tan(\theta) = \frac{y}{x}$, if $x \neq 0$ $\quad\quad$ $\cot(\theta) = \frac{x}{y}$, if $y \neq 0$

Based on these definitions, we have the Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\frac{x}{y} = \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}}$

Another two fundamental identities are the Quotient Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example: Finding Exact Values Using the Identities When Sine and Cosine Are Given.

Given $\sin\theta = \frac{\sqrt{5}}{5}$ and $\cos\theta = \frac{2\sqrt{5}}{5}$ find the exact values of the four remaining trig fxns of θ.

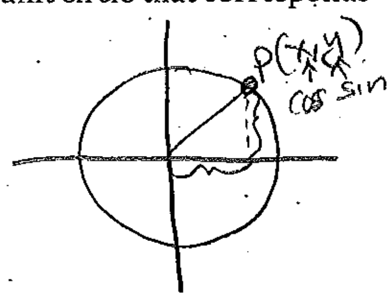
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}} = \frac{\sqrt{5}}{5}\cdot\frac{5}{2\sqrt{5}} = \frac{\cancel{5}\cancel{\sqrt{5}}}{2\cancel{\sqrt{5}}\cdot\cancel{5}} = \frac{1}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{2\sqrt{5}}{5}} = 1\cdot\frac{5}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\cancel{5}\sqrt{5}}{2\cdot\cancel{5}} = \frac{\sqrt{5}}{2}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\sqrt{5}/5} = \frac{5}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\cancel{5}\sqrt{5}}{\cancel{5}} = \sqrt{5}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{1/2} = 1\cdot\frac{2}{1} = 2$$

The equation of the unit circle is $y^2 + x^2 = 1$. If P = (x, y) is the point on the unit circle that corresponds to the angle θ, then $y = \sin\theta$ and $x = \cos\theta$. Therefore, we have....



$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

It is more customary to write it as....

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$\sin^2\theta = (\sin\theta)^2$

$\sin^2\theta \neq \sin\theta^2$

Using $sin^2(\theta) + cos^2(\theta) = 1$

a) divide both sides by $\sin^2\theta$:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

b) divide both sides by $\cos^2\theta$:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Collectively, the identities above are referred to as the **Pythagorean Identities**:

Fundamental Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Example: Finding the Exact Value of a Trig Expression Using Identities.

$\sec\theta = \frac{1}{\cos\theta}$

a) $\tan(20°) - \frac{\sin(20°)}{\cos(20°)}$

$\tan(20°) - \tan(20°)$

0

b) $sin^2\left(\frac{\pi}{12}\right) + \frac{1}{sec^2\left(\frac{\pi}{12}\right)}$

$\sin^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{\pi}{12}\right)$

1

$\frac{1}{\sec\theta} = \frac{1}{\frac{1}{\cos\theta}} = \cos\theta$

<u>FIND THE EXACT VALUES OF THE TRIG FXNS OF AN ANGLE GIVEN ONE OF THE FXNS AND THE QUADRANT OF THE ANGLE.</u>

When solving for sin θ or cos θ using the fundamental identities, we get....

$$\sin\theta = \pm\sqrt{1 - cos^2\theta}$$

and

$$\cos\theta = \pm\sqrt{1 - sin^2\theta}$$

$\sin^2\theta + \cancel{\cos^2\theta} = 1$

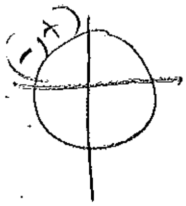
$\quad\; -\cos^2\theta \quad -\cos^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$

$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

To determine whether we use the positive or negative value depends on what quadrant the given trig fxn is in.

Example. Given that $\boxed{\sin\theta = \frac{1}{3}}$ and $\cos\theta < 0$, find the exact value of each of the remaining five trig fxns.



neg ⇒ Q II, ~~IV~~

$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1$

$\frac{1}{9} + \cos^2\theta = \frac{9}{9} - \frac{1}{9}$

$-\frac{1}{9}$

$\cos^2\theta = \frac{8}{9}$

$\cos\theta = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}}$

$\sqrt{8} = \sqrt{4}\cdot\sqrt{2} = 2\sqrt{2}$

$\boxed{\cos\theta = -\frac{2\sqrt{2}}{3}}$

$\tan\theta = \frac{1/3}{-2\sqrt{2}/3} = \frac{1}{-2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{-4}$

$\cot\theta = \frac{1}{\frac{\sqrt{2}}{-4}} = \frac{-4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{-4\sqrt{2}}{2} = -2\sqrt{2}$

$\sec\theta = \frac{1}{\frac{-2\sqrt{2}}{3}} = \frac{3}{-2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{-4}$

$\csc\theta = \frac{1}{\frac{1}{3}} = 1\cdot\frac{3}{1} = 3$

Example. Given that $\boxed{\tan\theta = \frac{1}{2}}$ and $\sin\theta < 0$, find the exact value of each of the remaining five trig fxns of θ.

Q III, ~~IV~~

① Since $\tan\theta$ is (+), and $\sin\theta$ is (−), then θ is in QIII

② Since $\tan\theta = \frac{1}{2} = \frac{y}{x}$, then $P(-2,-1)$ in QIII

③ $x^2 + y^2 = r^2 \Rightarrow (-2)^2 + (-1)^2 = r^2$

$4 + 1 = r^2 \Rightarrow r^2 = 5 \Rightarrow r = \sqrt{5}$

④ $\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \boxed{\frac{-\sqrt{5}}{5} = \sin\theta}$

$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \boxed{\frac{-2\sqrt{5}}{5} = \cos\theta}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{y}{r}} = \frac{r}{y} = \frac{\sqrt{5}}{-1} = -\sqrt{5}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{x}{r}} = \frac{r}{x} = \frac{\sqrt{5}}{-2}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{1}{2}} = 2$

5.3 Properties of the Trig Fxns PART II

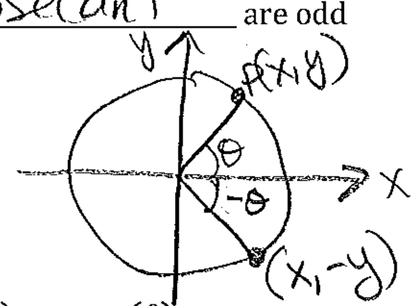
EVEN - ODD PROPERTIES TO FIND THE EXACT VALUES OF THE TRIG FXNS

Recall:

$$Even\ fxns\colon\ f(-\theta) = f(\theta)$$

$$Odd\ fxns\colon\ f(-\theta) = -f(\theta)$$

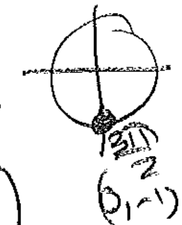
Sine, tangent, cotangent, and cosecant are odd functions. Cosine and secant are even functions.



Even – Odd Properties

$\sin(-\theta) = -\sin(\theta)$ $\cos(-\theta) = \cos(\theta)$ $\tan(-\theta) = -\tan(\theta)$

$\csc(-\theta) = -ccs(\theta)$ $\sec(-\theta) = \sec(\theta)$ $\cot(-\theta) = -\cot(\theta)$



Example. Find the exact value of

a) $\sin(-45°) = -\sin(45°)$
$= -\frac{\sqrt{2}}{2}$

c) $\cot\left(-\frac{3\pi}{2}\right) = -\cot\left(\frac{3\pi}{2}\right)$
$= -\frac{0}{-1} = 0$

b) $\cos(-\pi) = \cos(\pi)$
$= -1$

d) $\tan\left(-\frac{37\pi}{4}\right) = -\tan\left(\frac{37\pi}{4}\right)$
$= -\tan\left(\frac{\pi}{4} + \frac{36\pi}{4}\right)$
$= -\tan\left(\frac{\pi}{4}\right)$
$= -\frac{\sqrt{2}/2}{\sqrt{2}/2} = -1$

$\frac{37\pi}{4} - \frac{36\pi}{4} = \frac{\pi}{4}$