SUPPLEMENT TO §5.6

Graphing Sinsoidal Functions: Phase Shift vs. Horizontal Shift

Let's consider the function $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$. Using what we study in MTH 111 about graph transformations, it should be apparent that the graph of $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ can be obtained by transforming the graph of $f(x) = \sin(x)$. (To confirm this, notice that $g(x)$ can be expressed in terms of $f(x) = \sin(x)$ as $g(x) = f\left(2x - \frac{2\pi}{3}\right)$.) Since the constants "2" and "$\frac{2\pi}{3}$" are multiplied by and subtracted from the input variable, x, what we study in MTH 111 tells us that these constants represent a horizontal stretch/compression and a horizontal shift, respectively.

It is often recommended in MTH 111 that we factor-out the horizontal stretching/compressing factor before transforming the graph, i.e., it's often recommended that we first re-write $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ as $g(x) = \sin\left(2\left(x - \frac{\pi}{3}\right)\right)$. After writing g in this format, we can draw its graph by performing the following sequence of transformations of the "base function" $f(x) = \sin(x)$:

1st compress horizontally by a factor of $\frac{1}{2}$

2nd shift to the right $\frac{\pi}{3}$ units

The advantage of this method is that the y-intercept of $f(x) = \sin(x)$, $(0, 0)$, ends-up exactly where the horizontal shift suggests: when we compress the x-coordinate of $(0, 0)$ by a factor of $\frac{1}{2}$, it doesn't move since $\frac{1}{2} \cdot 0 = 0$; then, when we shift the graph right $\frac{\pi}{3}$ units, the point $(0, 0)$ ends up at $\left(\frac{\pi}{3}, 0\right)$; so the y-intercept ends up moving to right $\frac{\pi}{3}$ units, exactly how far we shifted.

Compare this with the alternative method: we can leave $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ as-is and skip factoring-out the horizontal stretching/compressing factor, but then we need the following sequence to transform $f(x) = \sin(x)$ into the graph of g:

1st shift to the right $\frac{2\pi}{3}$ units

2nd compress horizontally by a factor of $\frac{1}{2}$

The disadvantage of this method is that the y-intercept of $f(x) = \sin(x)$ **doesn't** end-up where the horizontal shift suggests: When we shift $(0, 0)$ to the right $\frac{2\pi}{3}$ units, it moves to

$\left(\frac{2\pi}{3}, 0\right)$; then, when we compress the x-coordinate of this point by a factor of $\frac{1}{2}$, it changes to $\frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$ and the point moves to $\left(\frac{\pi}{3}, 0\right)$ so the y-intercept **doesn't** end up shifted to right $\frac{2\pi}{3}$ units.

In Figure 7, we've graphed $y = g(x)$. Notice that this graph behaves like the graph of $f(x) = \sin(x)$ at $x = \frac{\pi}{3}$, i.e., $y = g(x)$ appears to have been shifted to the right $\frac{\pi}{3}$ units. For this reason, $\frac{\pi}{3}$ is called the ***horizontal shift*** of $g(x) = \sin\left(2x - \frac{2\pi}{3}\right) = \sin\left(2\left(x - \frac{\pi}{3}\right)\right)$.

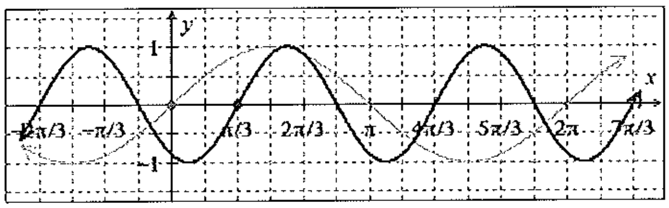


Figure 7: $y = g(x)$ with $f(x) = \sin(x)$.

The constant $\frac{2\pi}{3}$ is given a different name, ***phase shift***, since it can be used to determine how far "out-of-phase" a sinusoidal function is in comparison with $y = \sin(x)$ or $y = \cos(x)$. To determine how far out-of-phase a sinusoidal function is, we can determine the ratio of the phase shift and 2π. (We use 2π is because it's the period of $y = \sin(x)$ and $y = \cos(x)$.) Since $\frac{2\pi}{3}$ is the phase shift for $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$, the graph of $y = g(x)$ is out-of-phase $\frac{2\pi/3}{2\pi} = \frac{1}{3}$ of a period. (Since this number is positive, it represents a horizontal shift to the right $\frac{1}{3}$ of a period.)

Phase Shift vs. Horizontal Shift

Given a sinusoidal function of the form $y = A\sin(\omega x - C) + k$ or $y = A\cos(\omega x - C) + k$, the ***phase shift*** is C and $\frac{|C|}{2\pi}$ represents the fraction of a period that the graph has been shifted (shift to the right if C is positive or to the left if C is negative).

If we re-write the function as $y = A\sin\left(\omega\left(x - \frac{C}{\omega}\right)\right) + k$ or $y = A\cos\left(\omega\left(x - \frac{C}{\omega}\right)\right) + k$, we can see that the ***horizontal shift*** is $\frac{C}{\omega}$ units (shift to the right if $\frac{C}{\omega}$ is positive or to the left if $\frac{C}{\omega}$ is negative).

EXAMPLE 7: Identify the phase shift and horizontal shift of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$.

SOLUTION:

- The phase shift of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ is $\frac{\pi}{4}$. This tells us that the graph of $y = g(x)$ is out-of-phase $\frac{\left|\pi/4\right|}{2\pi} = \frac{1}{8}$ of a period, i.e., compared with $y = \cos(x)$, the graph of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ has been shifted one-eighth of a period to the right.

- To find the horizontal shift, we need to factor-out 3 from $3x - \frac{\pi}{4}$:

$$\begin{aligned} g(x) &= \cos\left(3x - \frac{\pi}{4}\right) \\ &= \cos\left(3\left(x - \frac{\pi}{3\cdot 4}\right)\right) \\ &= \cos\left(3\left(x - \frac{\pi}{12}\right)\right) \end{aligned}$$

So the horizontal shift is $\frac{\pi}{12}$. This tells us that, compared with $y = \cos(x)$, the graph of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ has been shifted $\frac{\pi}{12}$ units to the right.

Notice that the period of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ is $2\pi \cdot \frac{1}{3} = \frac{2\pi}{3}$, and that one-eighth of $\frac{2\pi}{3}$ is $\frac{2\pi}{3} \cdot \frac{1}{8} = \frac{\pi}{12}$, so a shift of one-eighth of a period is the same as a shift of $\frac{\pi}{12}$ units!

EXAMPLE 8: Draw a graph $q(t) = 2\sin(4t + \pi) + 1$. First, find its amplitude, period, midline, phase shift, and horizontal shift.

SOLUTION:

- Amplitude: $|A| = |2| = 2$
- Period: $P = 2\pi \cdot \frac{1}{|\omega|} = \frac{2\pi}{4} = \frac{\pi}{2}$
- Midline: $y = 1$
- Phase shift: $-\pi$ (this tells us that the graph is out-of-phase $\frac{|-\pi|}{2\pi} = \frac{1}{2}$ of a period)

- Horizontal shift: $\frac{\pi}{4}$ units to the left since:

$$\begin{aligned} q(t) &= 2\sin\left(4t + \pi\right) + 1 \\ &= 2\sin\left(4\left(t + \tfrac{\pi}{4}\right)\right) + 1 \\ &= 2\sin\left(4\left(t - \left(-\tfrac{\pi}{4}\right)\right)\right) + 1 \end{aligned}$$

Now we can draw a graph of $q(t) = 2\sin\left(4t + \pi\right) + 1$ by drawing a sinusoidal function with the necessary features; see Figure 8.

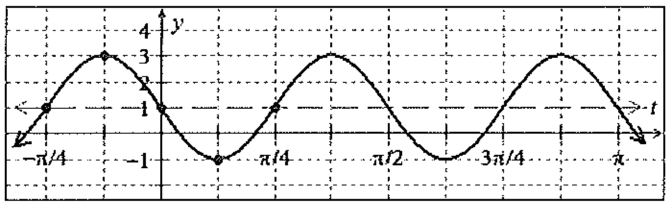


Figure 8: $y = q(t)$

EXERCISES:

1. Draw a graph of each of the following functions. List the amplitude, midline, period, phase shift, and horizontal shift.

a. $f(x) = 3\sin\left(3x - \frac{\pi}{2}\right)$

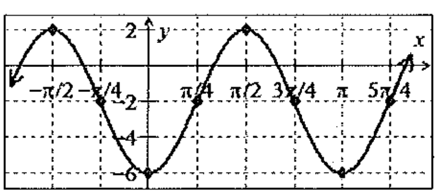
b. $g(t) = \cos\left(4t + \pi\right) + 3$

c. $m(\theta) = 2\cos(2\pi\theta - \pi) + 4$

d. $n(x) = -4\sin\left(\pi x + \frac{\pi}{4}\right) - 2$

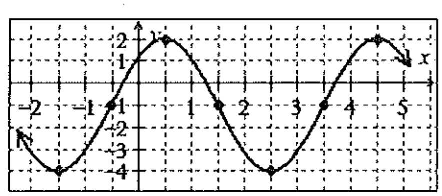
2. Find two algebraic rules (one involving sine and one involving cosine) for each of the functions graphed below.

a.



A graph of $y = p(t)$

b.



A graph of $y = q(x)$