5 Solve Equations Involving Inverse Trigonometric Functions

Equations that contain inverse trigonometric functions are called **inverse trigonometric equations.**

EXAMPLE 11 Solving an Equation Involving an Inverse Trigonometric Function

Solve the equation: $3 \sin^{-1} x = \pi$

Solution To solve an equation involving a single inverse trigonometric function, first isolate the inverse trigonometric function.

$$3 \sin^{-1} x = \pi$$

$$\sin^{-1} x = \frac{\pi}{3} \quad \text{Divide both sides by 3.}$$

$$x = \sin \frac{\pi}{3} \quad y = \sin^{-1} x \text{ means } x = \sin y.$$

$$x = \frac{\sqrt{3}}{2}$$

The solution set is $\left\{\frac{\sqrt{3}}{2}\right\}$.

Now Work PROBLEM 61

6.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. What are the domain and the range of $y = \sin x$? (p. 409)
2. A suitable restriction on the domain of the function $f(x) = (x - 1)^2$ to make it one-to-one would be ____. (pp. 289–290)
3. If the domain of a one-to-one function is $[3, \infty)$, the range of its inverse is ______. (pp. 285–286)
4. *True or False* The graph of $y = \cos x$ is decreasing on the interval $(0, \pi)$. (p. 423)
5. $\tan \frac{\pi}{4} =$ ____; $\sin \frac{\pi}{3} =$ ____ (pp. 395–398)
6. $\sin\left(-\frac{\pi}{6}\right) =$ ____; $\cos \pi =$ ________. (pp. 398–400)

Concepts and Vocabulary

7. $y = \sin^{-1} x$ means ________, where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
8. $\cos^{-1}(\cos x) = x$, where ________.
9. $\tan(\tan^{-1} x) = x$, where ____________.
10. *True or False* The domain of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
11. *True or False* $\sin(\sin^{-1} 0) = 0$ and $\cos(\cos^{-1} 0) = 0$.
12. *True or False* $y = \tan^{-1} x$ means $x = \tan y$, where $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Skill Building

In Problems 13–24, find the exact value of each expression.

13. $\sin^{-1} 0$
14. $\cos^{-1} 1$
15. $\sin^{-1}(-1)$
16. $\cos^{-1}(-1)$
17. $\tan^{-1} 0$
18. $\tan^{-1}(-1)$
19. $\sin^{-1} \frac{\sqrt{2}}{2}$
20. $\tan^{-1} \frac{\sqrt{3}}{3}$
21. $\tan^{-1} \sqrt{3}$
22. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
23. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
24. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

In Problems 25–36, use a calculator to find the value of each expression rounded to two decimal places.

25. $\sin^{-1} 0.1$
26. $\cos^{-1} 0.6$
27. $\tan^{-1} 5$
28. $\tan^{-1} 0.2$
29. $\cos^{-1} \frac{7}{8}$
30. $\sin^{-1} \frac{1}{8}$
31. $\tan^{-1}(-0.4)$
32. $\tan^{-1}(-3)$
33. $\sin^{-1}(-0.12)$
34. $\cos^{-1}(-0.44)$
35. $\cos^{-1} \frac{\sqrt{2}}{3}$
36. $\sin^{-1} \frac{\sqrt{3}}{5}$

In Problems 37–44, find the exact value of each expression. Do not use a calculator.

37. $\cos^{-1}\left(\cos\frac{4\pi}{5}\right)$
38. $\sin^{-1}\left[\sin\left(-\frac{\pi}{10}\right)\right]$
39. $\tan^{-1}\left[\tan\left(-\frac{3\pi}{8}\right)\right]$
40. $\sin^{-1}\left[\sin\left(-\frac{3\pi}{7}\right)\right]$
41. $\sin^{-1}\left(\sin\frac{9\pi}{8}\right)$
42. $\cos^{-1}\left[\cos\left(-\frac{5\pi}{3}\right)\right]$
43. $\tan^{-1}\left(\tan\frac{4\pi}{5}\right)$
44. $\tan^{-1}\left[\tan\left(-\frac{2\pi}{3}\right)\right]$

In Problems 45–52, find the exact value, if any, of each composite function. If there is no value, say it is "not defined." Do not use a calculator.

45. $\sin\left(\sin^{-1}\frac{1}{4}\right)$
46. $\cos\left[\cos^{-1}\left(-\frac{2}{3}\right)\right]$
47. $\tan(\tan^{-1}4)$
48. $\tan[\tan^{-1}(-2)]$
49. $\cos(\cos^{-1}1.2)$
50. $\sin[\sin^{-1}(-2)]$
51. $\tan(\tan^{-1}\pi)$
52. $\sin[\sin^{-1}(-1.5)]$

In Problems 53–60, find the inverse function f^{-1} of each function f. Find the range of f and the domain and range of f^{-1}.

53. $f(x) = 5\sin x + 2; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
54. $f(x) = 2\tan x - 3; -\frac{\pi}{2} < x < \frac{\pi}{2}$
55. $f(x) = -2\cos(3x); 0 \le x \le \frac{\pi}{3}$
56. $f(x) = 3\sin(2x); -\frac{\pi}{4} \le x \le \frac{\pi}{4}$
57. $f(x) = -\tan(x+1) - 3; -1 - \frac{\pi}{2} < x < \frac{\pi}{2} - 1$
58. $f(x) = \cos(x+2) + 1; -2 \le x \le \pi - 2$
59. $f(x) = 3\sin(2x+1); -\frac{1}{2} - \frac{\pi}{4} \le x \le -\frac{1}{2} + \frac{\pi}{4}$
60. $f(x) = 2\cos(3x+2); -\frac{2}{3} \le x \le -\frac{2}{3} + \frac{\pi}{3}$

In Problems 61–68, find the exact solution of each equation.

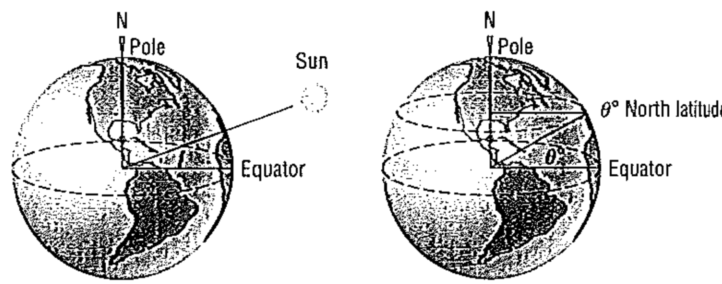
61. $4\sin^{-1}x = \pi$
62. $2\cos^{-1}x = \pi$
63. $3\cos^{-1}(2x) = 2\pi$
64. $-6\sin^{-1}(3x) = \pi$
65. $3\tan^{-1}x = \pi$
66. $-4\tan^{-1}x = \pi$
67. $4\cos^{-1}x - 2\pi = 2\cos^{-1}x$
68. $5\sin^{-1}x - 2\pi = 2\sin^{-1}x - 3\pi$

Applications and Extensions

In Problems 69–74, use the following discussion. The formula

$$D = 24\left[1 - \frac{\cos^{-1}(\tan i \tan \theta)}{\pi}\right]$$

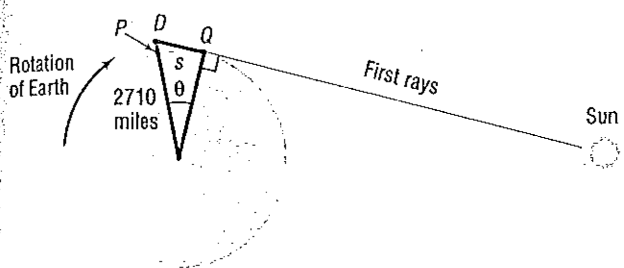
can be used to approximate the number of hours of daylight D when the declination of the Sun is $i°$ at a location $\theta°$ north latitude for any date between the vernal equinox and autumnal equinox. The declination of the Sun is defined as the angle i between the equatorial plane and any ray of light from the Sun. The latitude of a location is the angle θ between the Equator and the location on the surface of Earth, with the vertex of the angle located at the center of Earth. See the figure. To use the formula, $\cos^{-1}(\tan i \tan \theta)$ must be expressed in radians.



69. Approximate the number of hours of daylight in Houston, Texas (29°45′ north latitude), for the following dates:
    (a) Summer solstice ($i = 23.5°$)
    (b) Vernal equinox ($i = 0°$)
    (c) July 4 ($i = 22°48'$)
70. Approximate the number of hours of daylight in New York, New York (40°45′ north latitude), for the following dates:
    (a) Summer solstice ($i = 23.5°$)
    (b) Vernal equinox ($i = 0°$)
    (c) July 4 ($i = 22°48'$)
71. Approximate the number of hours of daylight in Honolulu, Hawaii (21°18′ north latitude), for the following dates:
    (a) Summer solstice ($i = 23.5°$)
    (b) Vernal equinox ($i = 0°$)
    (c) July 4 ($i = 22°48'$)
72. Approximate the number of hours of daylight in Anchorage, Alaska (61°10′ north latitude), for the following dates:
    (a) Summer solstice ($i = 23.5°$)
    (b) Vernal equinox ($i = 0°$)
    (c) July 4 ($i = 22°48'$)
73. Approximate the number of hours of daylight at the Equator (0° north latitude) for the following dates:
    (a) Summer solstice ($i = 23.5°$)
    (b) Vernal equinox ($i = 0°$)
    (c) July 4 ($i = 22°48'$)
    (d) What do you conclude about the number of hours of daylight throughout the year for a location at the Equator?
74. Approximate the number of hours of daylight for any location that is 66°30′ north latitude for the following dates:
    (a) Summer solstice ($i = 23.5°$)
    (b) Vernal equinox ($i = 0°$)
    (c) July 4 ($i = 22°48'$)

(d) Thanks to the symmetry of the orbital path of Earth around the Sun, the number of hours of daylight on the winter solstice may be found by computing the number of hours of daylight on the summer solstice and subtracting this result from 24 hours. Compute the number of hours of daylight for this location on the winter solstice. What do you conclude about daylight for a location at 66°30′ north latitude?

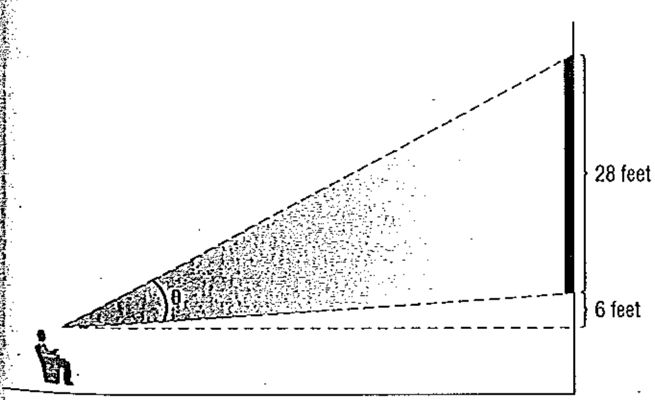
75. Being the First to See the Rising Sun Cadillac Mountain, elevation 1530 feet, is located in Acadia National Park, Maine, and is the highest peak on the east coast of the United States. It is said that a person standing on the summit will be the first person in the United States to see the rays of the rising Sun. How much sooner would a person atop Cadillac Mountain see the first rays than a person standing below, at sea level?



Hint: Consult the figure. When the person at D sees the first rays of the Sun, the person at P does not. The person at P sees the first rays of the Sun only after Earth has rotated so that P is at location Q. Compute the length of the arc subtended by the central angle θ. Then use the fact that at the latitude of Cadillac Mountain, in 24 hours a length of $2\pi(2710) \approx 17{,}027.4$ miles is subtended, and find the time that it takes to subtend this length.

76. Movie Theater Screens Suppose that a movie theater has a screen that is 28 feet tall. When you sit down, the bottom of the screen is 6 feet above your eye level. The angle formed by drawing a line from your eye to the bottom of the screen and another line from your eye to the top of the screen is called the **viewing angle.** In the figure, θ is the viewing angle. Suppose that you sit x feet from the screen. The viewing angle θ is given by the function

$$\theta(x) = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right)$$



(a) What is your viewing angle if you sit 10 feet from the screen? 15 feet? 20 feet?

(b) If there is 5 feet between the screen and the first row of seats and there is 3 feet between each row and the row behind it, which row results in the largest viewing angle?

(c) Using a graphing utility, graph

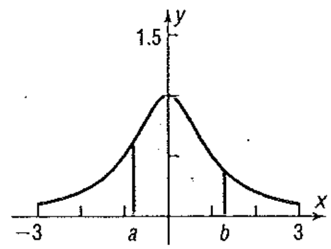
$$\theta(x) = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right)$$

What value of x results in the largest viewing angle?

77. Area under a Curve The area under the graph of $y = \dfrac{1}{1+x^2}$ and above the x-axis between $x = a$ and $x = b$ is given by

$$\tan^{-1} b - \tan^{-1} a$$

See the figure.



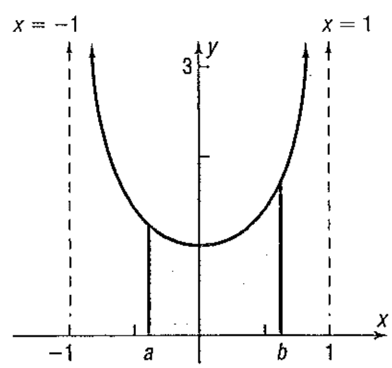
(a) Find the exact area under the graph of $y = \dfrac{1}{1+x^2}$ and above the x-axis between $x = 0$ and $x = \sqrt{3}$.

(b) Find the exact area under the graph of $y = \dfrac{1}{1+x^2}$ and above the x-axis between $x = -\dfrac{\sqrt{3}}{3}$ and $x = 1$.

78. Area under a Curve The area under the graph of $y = \dfrac{1}{\sqrt{1-x^2}}$ and above the x-axis between $x = a$ and $x = b$ is given by

$$\sin^{-1} b - \sin^{-1} a$$

See the figure.



(a) Find the exact area under the graph of $y = \dfrac{1}{\sqrt{1-x^2}}$ and above the x-axis between $x = 0$ and $x = \dfrac{\sqrt{3}}{2}$.

(b) Find the exact area under the graph of $y = \dfrac{1}{\sqrt{1-x^2}}$ and above the x-axis between $x = -\dfrac{1}{2}$ and $x = \dfrac{1}{2}$.

Problems 79 and 80 require the following discussion:

The shortest distance between two points on Earth's surface can be determined from the latitude and longitude of the two locations. For example, if location 1 has $(\text{lat}, \text{lon}) = (\alpha_1, \beta_1)$ *and location 2 has* $(\text{lat}, \text{lon}) = (\alpha_2, \beta_2)$, *the shortest distance between the two locations is approximately* $d = r\cos^{-1}[(\cos\alpha_1\cos\beta_1\cos\alpha_2\cos\beta_2) + (\cos\alpha_1\sin\beta_1\cos\alpha_2\sin\beta_2) + (\sin\alpha_1\sin\alpha_2)]$, *where* r = *radius of Earth* $\approx$ *3960 miles and the inverse cosine function is expressed in radians. Also, N latitude and E longitude are positive angles, and S latitude and W longitude are negative angles.*

City	Latitude	Longitude
Chicago, IL	41°50′N	87°37′W
Honolulu, HI	21°18′N	157°50′W
Melbourne, Australia	37°47′S	144°58′E

Source: www.infoplease.com

79. Shortest Distance from Chicago to Honolulu Find the shortest distance from Chicago, latitude 41°50′N, longitude 87°37′W to Honolulu, latitude 21°18′N, longitude 157°50′W. Round your answer to the nearest mile.

80. Shortest Distance from Honolulu to Melbourne, Australia Find the shortest distance from Honolulu to Melbourne, Australia, latitude 37°47′S, longitude 144°58′E. Round your answer to the nearest mile.

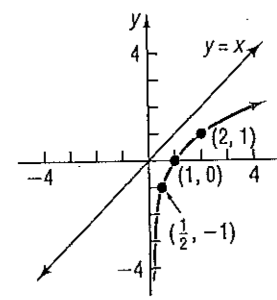
Retain Your Knowledge

Problems 81–84 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

81. Solve: $|3x - 2| + 5 \le 9$

82. State why the graph of the function shown to the right is one-to-one. Then draw the graph of the inverse function f^{-1}.

Hint: The graph of $y = x$ is given.



83. The exponential function $f(x) = 1 + 2^x$ is one-to-one. Find f^{-1}.

84. Factor: $(2x+1)^{-\frac{1}{2}}(x^2+3)^{-\frac{1}{2}} - (x^2+3)^{-\frac{3}{2}}x(2x+1)^{\frac{1}{2}}$

'Are You Prepared?' Answers

1. Domain: the set of all real numbers; Range: $\{y \mid -1 \le y \le 1\}$

2. Two answers are possible: $x \le 1$ or $x \ge 1$

3. $[3, \infty)$

4. True

5. $1; \dfrac{\sqrt{3}}{2}$

6. $-\dfrac{1}{2}; -1$

6.2 The Inverse Trigonometric Functions (Continued)

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Finding Exact Values Given the Value of a Trigonometric Function and the Quadrant of the Angle (Section 5.3, pp. 413–416)
- Graphs of the Secant, Cosecant, and Cotangent Functions (Section 5.5, pp. 438–441)
- Domain and Range of the Secant, Cosecant, and Cotangent Functions (Section 5.3, pp. 407–409)
- Use a Circle of Radius r to Evaluate Trigonometric Functions (pp. 400–401)

Now Work the 'Are You Prepared?' problems on page 480.

OBJECTIVES

1 Find the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions (p. 477)

2 Define the Inverse Secant, Cosecant, and Cotangent Functions (p. 478)

3 Use a Calculator to Evaluate $\sec^{-1}x$, $\csc^{-1}x$, and $\cot^{-1}x$ (p. 479)

4 Write a Trigonometric Expression as an Algebraic Expression (p. 479)