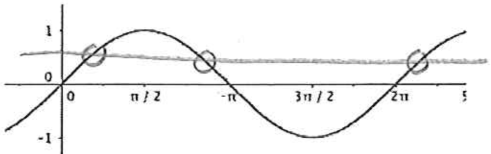
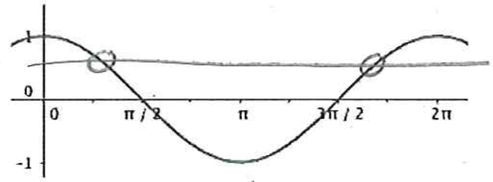
6.1 THE INVERSE SINE, COSINE, AND TANGENT FXNS

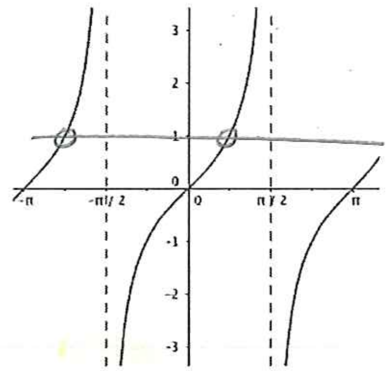
Sine, cosine, and tangent are not one-to-one functions; they fail the horizontal line test. Recall for $0 \le \theta \le 2\pi$ on the unit circle, sine and cosine do not have unique outputs. For example, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.



y = sin(x)



y = cos(x)



y = tan(x)

THE INVERSE SINE

However, if we restrict the domain of the sinusoidal fxns, the restricted fxns are now one-to-one fxns and have inverse fxns.

If y = sin(x) is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.... [show graph]

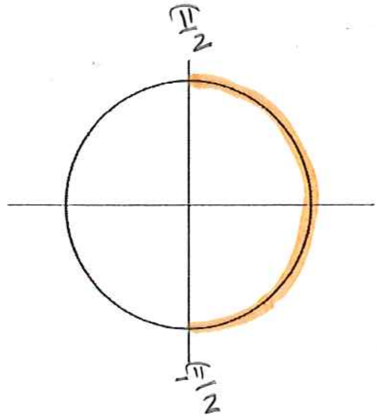
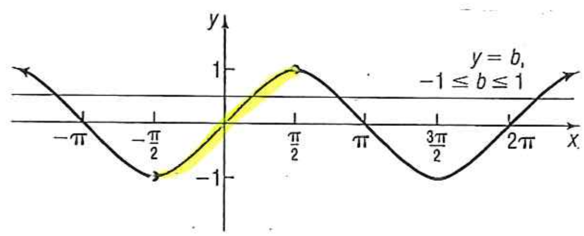
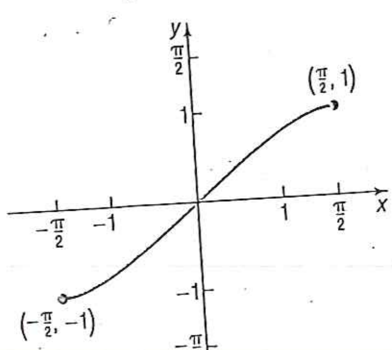


Figure 1
$y = \sin x, -\infty < x < \infty,$
$-1 \le y \le 1$

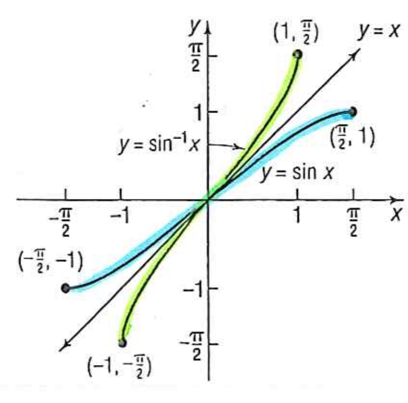


So, the graph of y = sin(x), $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ looks like

Figure 2
$y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}, -1 \le y \le 1$



And now has an inverse function that is reflected over the line y = x.



The equation for the inverse of y = f(x) = sin(x) is obtained by interchanging __x__ and __y__. Therefore, the inverse function of y = sin(x) is __$y = \sin^{-1}(x)$__ where __$-\frac{\pi}{2}$__ $\leq y \leq$ __$\frac{\pi}{2}$__. Using fxn notation, if f(x) = sin(x), then $f^{-1}(x) =$ __$\sin^{-1}(x)$__.

By definition

$$y = sin^{-1}(x) \quad means \quad x = \sin(y)$$

$$where -1 \leq x \leq 1 \qquad and \qquad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$y = sin^{-1}(x)$ is read as "__y is the angle or real number where sine equals x__"

or "__y is the inverse sine of x__".

Therefore, the output of $sin^{-1}(x)$ is the angle that corresponds to the sine value, x.

Example. Find the exact value of ***a)*** $sin^{-1}(1)$ ***b)*** $sin^{-1}\left(-\frac{1}{2}\right)$.

a) let $\theta = \sin^{-1}(1)$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\theta = \sin(\sin^{-1} 1)$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$ $\therefore \sin^{-1}(1) = \frac{\pi}{2}$

Example. Find an approximate value of ***a)*** $sin^{-1}\left(\frac{1}{3}\right)$ ***b)*** $sin^{-1}\left(-\frac{1}{4}\right)$. Express the answer in radians rounded to two decimal places. [Use a calculator, set mode to radians].

a) $\sin^{-1}\left(\frac{1}{3}\right) \approx .34$

b) $\sin^{-1}\left(\frac{1}{4}\right) \approx .25$

USE PROPERTIES OF INVERSE FXNS TO FIND EXACT VALUES OF CERTAIN COMPOSITE FXNS.

Recall that $f^{-1}(f(x)) = x$ for all x in the domain of f and that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. In terms of the sine fxn and its inverse, these properties are of the form

$$f^{-1}(f(x)) = sin^{-1}(\sin x) = x \quad where -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \quad \text{(A)}$$

$$f(f^{-1}(x)) = sin(sin^{-1} x) = x \quad where -1 \leq x \leq 1 \quad \text{(B)}$$

Example. Find the exact value of each of the following composite fxns.

a) $sin^{-1}\left(sin\frac{\pi}{8}\right)$

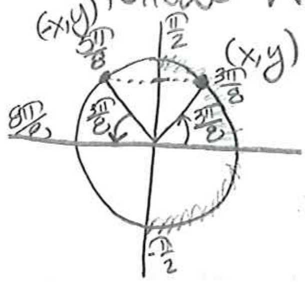
follows A

$\sin\frac{\pi}{8}$ is within $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

so

$\sin^{-1}\left(\sin\frac{\pi}{8}\right)=\frac{\pi}{8}$

b) $sin^{-1}\left(sin\frac{5\pi}{8}\right)$



follows "A" : notice $\frac{5\pi}{8}$ is not in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

but $\sin\left(\frac{5\pi}{8}\right)=\sin\left(\frac{3\pi}{8}\right)$

$\therefore \sin^{-1}\left(\sin\frac{5\pi}{8}\right)=\sin^{-1}\left(\sin\frac{3\pi}{8}\right)$

and $\sin^{-1}\left(\sin\frac{3\pi}{8}\right)=\frac{3\pi}{8}$

$\frac{3\pi}{8}$ is within $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Example. Find the exact value, if any, of each composite function.

a) $sin(sin^{-1}0.8)$

follows "B"

Is 0.8 within $[-1,1]$? yes!

$\therefore \sin(\sin^{-1}0.8)=0.8$

b) $sin(sin^{-1}1.8)$

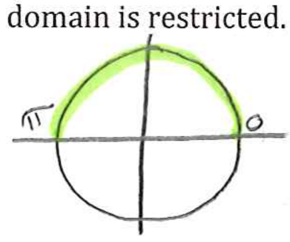
follows "B"

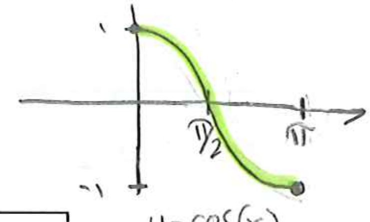
Is 1.8 within $[-1,1]$? NO!

$\therefore$ not defined

THE INVERSE COSINE

Similar to sine, cosine is not a one-to-one function unless the domain is restricted. Restricting y = cos(x) to $0\le x\le\pi$, it now has an inverse function.





y = cos(x)
$0\le x\le\pi$

By definition

$$y = cos^{-1}(x) \;\; means \;\; x = \cos(y)$$
$$where -1 \le x \le 1 \quad and \quad 0 \le y \le \pi$$

The output of $cos^{-1}(x)$ is the angle that corresponds to the cosine value, x.

Example. Find the exact value of:

a) $cos^{-1}(0)$

Let $\theta=\cos^{-1}(0)$ $[0,\pi]$

$\cos(\theta)=\cancel{\cos}\left[\cancel{\cos^{-1}}(0)\right]$ take cosine of both sides

$\cos\theta=0$ $[0,\pi]$

$\theta=\frac{\pi}{2}$

b) $cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Let $\theta=\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ for $0\le x\le\pi$

$\cos\theta=\cos\left(\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$ $[0,\pi]$

$\cos\theta=-\frac{\sqrt{2}}{2}$ $[0,\pi]$

$\theta=\frac{3\pi}{4}$ $\therefore \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4}$

For the cosine fxn and its inverse, the following properties hold.

$$f^{-1}(f(x)) = cos^{-1}(\cos x) = x \quad where \quad 0 \le x \le \pi \quad \text{(A)}$$

$$f(f^{-1}(x)) = cos(cos^{-1} x) = x \quad where -1 \le x \le 1 \quad \text{(B)}$$

Example. Find the exact value of:

a) $cos^{-1}\left(cos\frac{\pi}{12}\right)$

Follows "A"

is $\frac{\pi}{12}$ within $[0,\pi]$?

yes!

$\therefore cos^{-1}(cos\frac{\pi}{12}) = \frac{\pi}{12}$

b) $\cos\,[cos^{-1}(-0.4)]$

Follows "B"

Is -0.4 within $[-1,1]$?

yes

$\therefore cos[cos^{-1}(-0.4)] = -0.4$

c) $cos^{-1}\left[cos\left(-\frac{2\pi}{3}\right)\right]$

Follows 'A'

is $-\frac{2\pi}{3}$ within $[0,\pi]$?

No

However, cosine is an even fxn

$\therefore cos(-\frac{2\pi}{3}) = cos(\frac{2\pi}{3})$

$\frac{2\pi}{3}$ is within $[0,\pi]$

so

$cos^{-1}[cos(-\frac{2\pi}{3})] =$

$cos^{-1}[cos(\frac{2\pi}{3})] =$

$\frac{2\pi}{3}$

d) $\cos\,(cos^{-1}\pi)$

Follows "B"

is π within $[-1,1]$?

No.

$\therefore$ not defined

THE INVERSE TANGENT FXN

Tangent is not one-to-one. However, by restricting the domain to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, y = tan(x) is one-to-one and has an inverse function.

$(-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow$ end points NOT included

By definition

$$y = tan^{-1}(x) \quad means \quad x = \tan(y)$$

$$where -\infty < x < \infty \qquad and \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The output of $tan^{-1}(x)$ is the angle that corresponds to the tan value, x.

Example. Find the exact value of :

a) $tan^{-1} 1$

Let $\theta = \tan^{-1}(1)$

$\theta = \tan^{-1}(1) \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\tan(\theta) = \tan(\tan^{-1}(1)) \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\tan(\theta) = 1$

$\theta = \frac{\pi}{4}$ $\quad \left[\text{Remember: } \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1\right]$

$\therefore \tan^{-1}(1) = \frac{\pi}{4}$

b) $tan^{-1}(-\sqrt{3})$

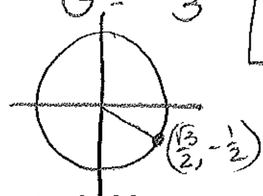
Let $\theta = \tan^{-1}(-\sqrt{3})$

$\theta = \tan^{-1}(-\sqrt{3}) \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\tan(\theta) = \tan(\tan^{-1}(-\sqrt{3})) \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\tan(\theta) = -\sqrt{3} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\theta = -\frac{\pi}{3}$ $\quad \left[\text{Remember: } \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}\right]$



$\therefore \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

For the tangent fxn and its inverse, the following properties hold.

$f^{-1}(f(x)) = tan^{-1}(\tan x) = x$	$where -\frac{\pi}{2} < x < \frac{\pi}{2}$	(A)
$f(f^{-1}(x)) = tan(tan^{-1} x) = x$	$where -\infty \leq x \leq \infty$	(B)

<u>SOLVE EQUATIONS INVOLVING INVERSE TRIG FXNS</u>

When solving an equation involving a single inverse trig fxn, first isolate the inverse trig fxn.

Example. Solve the equation: $3sin^{-1}x = \pi$

$\frac{\cancel{3}\sin^{-1}(x)}{\cancel{3}} = \frac{\pi}{3}$

$\sin^{-1}(x) = \frac{\pi}{3} \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\cancel{\sin}[\cancel{\sin^{-1}}(x)] = \sin(\frac{\pi}{3})$

$x = \sin(\frac{\pi}{3}) \qquad -1 \leq x \leq 1$

$x = \frac{\sqrt{3}}{2}$