SUMMARY

Sum and Difference Formulas

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

6.5 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The distance d from the point $(2,-3)$ to the point $(5,1)$ is ____. (pp. 3–5)
2. If $\sin\theta=\frac{4}{5}$ and θ is in quadrant II, then $\cos\theta=$ ____. (pp. 413–416)
3. (a) $\sin\frac{\pi}{4}\cdot\cos\frac{\pi}{3}=$ ____. (pp. 395–398)
   (b) $\tan\frac{\pi}{4}-\sin\frac{\pi}{6}=$ ____. (pp. 395–398)
4. If $\sin\alpha=-\frac{4}{5},\ \pi<\alpha<\frac{3\pi}{2}$, then $\cos\alpha=$ ____. (pp. 413–416)

Concepts and Vocabulary

5. $\cos(\alpha+\beta)=\cos\alpha\cos\beta$ ___ $\sin\alpha\sin\beta$
6. $\sin(\alpha-\beta)=\sin\alpha\cos\beta$ ___ $\cos\alpha\sin\beta$
7. *True or False* $\sin(\alpha+\beta)=\sin\alpha+\sin\beta+2\sin\alpha\sin\beta$
8. *True or False* $\tan 75°=\tan 30°+\tan 45°$
9. *True or False* $\cos\left(\frac{\pi}{2}-\theta\right)=\cos\theta$
10. *True or False* If $f(x)=\sin x$ and $g(x)=\cos x$, then $g(\alpha+\beta)=g(\alpha)g(\beta)-f(\alpha)f(\beta)$

Skill Building

In Problems 11–22, find the exact value of each expression.

11. $\sin\frac{5\pi}{12}$
12. $\sin\frac{\pi}{12}$
13. $\cos\frac{7\pi}{12}$
14. $\tan\frac{7\pi}{12}$
15. $\cos 165°$
16. $\sin 105°$
17. $\tan 15°$
18. $\tan 195°$
19. $\sin\frac{17\pi}{12}$
20. $\tan\frac{19\pi}{12}$
21. $\sec\left(-\frac{\pi}{12}\right)$
22. $\cot\left(-\frac{5\pi}{12}\right)$

In Problems 23–32, find the exact value of each expression.

23. $\sin 20°\cos 10°+\cos 20°\sin 10°$
24. $\sin 20°\cos 80°-\cos 20°\sin 80°$
25. $\cos 70°\cos 20°-\sin 70°\sin 20°$
26. $\cos 40°\cos 10°+\sin 40°\sin 10°$
27. $\frac{\tan 20°+\tan 25°}{1-\tan 20°\tan 25°}$
28. $\frac{\tan 40°-\tan 10°}{1+\tan 40°\tan 10°}$
29. $\sin\frac{\pi}{12}\cos\frac{7\pi}{12}-\cos\frac{\pi}{12}\sin\frac{7\pi}{12}$
30. $\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}-\sin\frac{5\pi}{12}\sin\frac{7\pi}{12}$
31. $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}+\sin\frac{5\pi}{12}\sin\frac{\pi}{12}$
32. $\sin\frac{\pi}{18}\cos\frac{5\pi}{18}+\cos\frac{\pi}{18}\sin\frac{5\pi}{18}$

In Problems 33–38 find the exact value of each of the following under the given conditions:

(a) $\sin(\alpha+\beta)$ *(b)* $\cos(\alpha+\beta)$ *(c)* $\sin(\alpha-\beta)$ *(d)* $\tan(\alpha-\beta)$

33. $\sin\alpha=\frac{3}{5},\ 0<\alpha<\frac{\pi}{2};\quad \cos\beta=\frac{2\sqrt{5}}{5},\ -\frac{\pi}{2}<\beta<0$
34. $\cos\alpha=\frac{\sqrt{5}}{5},\ 0<\alpha<\frac{\pi}{2};\quad \sin\beta=-\frac{4}{5},\ -\frac{\pi}{2}<\beta<0$
35. $\tan\alpha=-\frac{4}{3},\ \frac{\pi}{2}<\alpha<\pi;\quad \cos\beta=\frac{1}{2},\ 0<\beta<\frac{\pi}{2}$
36. $\tan\alpha=\frac{5}{12},\ \pi<\alpha<\frac{3\pi}{2};\quad \sin\beta=-\frac{1}{2},\ \pi<\beta<\frac{3\pi}{2}$
37. $\sin\alpha=\frac{5}{13},\ -\frac{3\pi}{2}<\alpha<-\pi;\quad \tan\beta=-\sqrt{3},\ \frac{\pi}{2}<\beta<\pi$
38. $\cos\alpha=\frac{1}{2},\ -\frac{\pi}{2}<\alpha<0;\quad \sin\beta=\frac{1}{3},\ 0<\beta<\frac{\pi}{2}$

39. If $\sin\theta = \frac{1}{3}$, θ in quadrant II, find the exact value of:

(a) $\cos\theta$ (b) $\sin\left(\theta + \frac{\pi}{6}\right)$

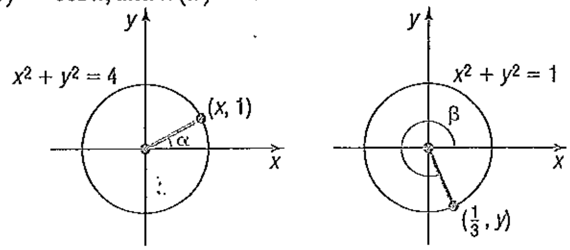
(c) $\cos\left(\theta - \frac{\pi}{3}\right)$ (d) $\tan\left(\theta + \frac{\pi}{4}\right)$

40. If $\cos\theta = \frac{1}{4}$, θ in quadrant IV, find the exact value of:

(a) $\sin\theta$ (b) $\sin\left(\theta - \frac{\pi}{6}\right)$

(c) $\cos\left(\theta + \frac{\pi}{3}\right)$ (d) $\tan\left(\theta - \frac{\pi}{4}\right)$

In Problems 41–46, use the figures to evaluate each function if $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = \tan x$.



41. $f(\alpha + \beta)$

42. $g(\alpha + \beta)$

43. $g(\alpha - \beta)$

44. $f(\alpha - \beta)$

45. $h(\alpha + \beta)$

46. $h(\alpha - \beta)$

In Problems 47–72, establish each identity.

47. $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

48. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

49. $\sin(\pi - \theta) = \sin\theta$

50. $\cos(\pi - \theta) = -\cos\theta$

51. $\sin(\pi + \theta) = -\sin\theta$

52. $\cos(\pi + \theta) = -\cos\theta$

53. $\tan(\pi - \theta) = -\tan\theta$

54. $\tan(2\pi - \theta) = -\tan\theta$

55. $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$

56. $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

57. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$

58. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$

59. $\dfrac{\sin(\alpha + \beta)}{\sin\alpha\cos\beta} = 1 + \cot\alpha\tan\beta$

60. $\dfrac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

61. $\dfrac{\cos(\alpha + \beta)}{\cos\alpha\cos\beta} = 1 - \tan\alpha\tan\beta$

62. $\dfrac{\cos(\alpha - \beta)}{\sin\alpha\cos\beta} = \cot\alpha + \tan\beta$

63. $\dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta}$

64. $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1 - \tan\alpha\tan\beta}{1 + \tan\alpha\tan\beta}$

65. $\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}$

66. $\cot(\alpha - \beta) = \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$

67. $\sec(\alpha + \beta) = \dfrac{\csc\alpha\csc\beta}{\cot\alpha\cot\beta - 1}$

68. $\sec(\alpha - \beta) = \dfrac{\sec\alpha\sec\beta}{1 + \tan\alpha\tan\beta}$

69. $\sin(\alpha - \beta)\sin(\alpha + \beta) = \sin^2\alpha - \sin^2\beta$

70. $\cos(\alpha - \beta)\cos(\alpha + \beta) = \cos^2\alpha - \sin^2\beta$

71. $\sin(\theta + k\pi) = (-1)^k\sin\theta$, k any integer

72. $\cos(\theta + k\pi) = (-1)^k\cos\theta$, k any integer

In Problems 73–84, find the exact value of each expression.

73. $\sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}0\right)$

74. $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}1\right)$

75. $\sin\left[\sin^{-1}\frac{3}{5} - \cos^{-1}\left(-\frac{4}{5}\right)\right]$

76. $\sin\left[\sin^{-1}\left(-\frac{4}{5}\right) - \tan^{-1}\frac{3}{4}\right]$

77. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{5}{13}\right)$

78. $\cos\left[\tan^{-1}\frac{5}{12} - \sin^{-1}\left(-\frac{3}{5}\right)\right]$

79. $\cos\left(\sin^{-1}\frac{5}{13} - \tan^{-1}\frac{3}{4}\right)$

80. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{12}{13}\right)$

81. $\tan\left(\sin^{-1}\frac{3}{5} + \frac{\pi}{6}\right)$

82. $\tan\left(\frac{\pi}{4} - \cos^{-1}\frac{3}{5}\right)$

83. $\tan\left(\sin^{-1}\frac{4}{5} + \cos^{-1}1\right)$

84. $\tan\left(\cos^{-1}\frac{4}{5} + \sin^{-1}1\right)$

In Problems 85–90, write each trigonometric expression as an algebraic expression containing u and v. Give the restrictions required on u and v.

85. $\cos(\cos^{-1} u + \sin^{-1} v)$
86. $\sin(\sin^{-1} u - \cos^{-1} v)$
87. $\sin(\tan^{-1} u - \sin^{-1} v)$
88. $\cos(\tan^{-1} u + \tan^{-1} v)$
89. $\tan(\sin^{-1} u - \cos^{-1} v)$
90. $\sec(\tan^{-1} u + \cos^{-1} v)$

In Problems 91–96, solve each equation on the interval $0 \le \theta < 2\pi$.

91. $\sin\theta - \sqrt{3}\cos\theta = 1$
92. $\sqrt{3}\sin\theta + \cos\theta = 1$
93. $\sin\theta + \cos\theta = \sqrt{2}$
94. $\sin\theta - \cos\theta = -\sqrt{2}$
95. $\tan\theta + \sqrt{3} = \sec\theta$
96. $\cot\theta + \csc\theta = -\sqrt{3}$

Applications and Extensions

97. Show that $\sin^{-1} v + \cos^{-1} v = \dfrac{\pi}{2}$.

98. Show that $\tan^{-1} v + \cot^{-1} v = \dfrac{\pi}{2}$.

99. Show that $\tan^{-1}\left(\dfrac{1}{v}\right) = \dfrac{\pi}{2} - \tan^{-1} v$, if $v > 0$.

100. Show that $\cot^{-1} e^{v} = \tan^{-1} e^{-v}$.

101. Show that $\sin(\sin^{-1} v + \cos^{-1} v) = 1$.

102. Show that $\cos(\sin^{-1} v + \cos^{-1} v) = 0$.

103. **Calculus** Show that the difference quotient for $f(x) = \sin x$ is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h} = \cos x \cdot \frac{\sin h}{h} - \sin x \cdot \frac{1 - \cos h}{h}$$

104. **Calculus** Show that the difference quotient for $f(x) = \cos x$ is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos x}{h} = -\sin x \cdot \frac{\sin h}{h} - \cos x \cdot \frac{1 - \cos h}{h}$$

105. **One, Two, Three**
   (a) Show that $\tan(\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3) = 0$.
   (b) Conclude from part (a) that

$$\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$$

Source: College Mathematics Journal, Vol. 37, No. 3, May 2006

106. **Electric Power** In an alternating current (ac) circuit, the instantaneous power p at time t is given by

$$p(t) = V_m I_m \cos\phi \sin^2(\omega t) - V_m I_m \sin\phi \sin(\omega t)\cos(\omega t)$$

Show that this is equivalent to

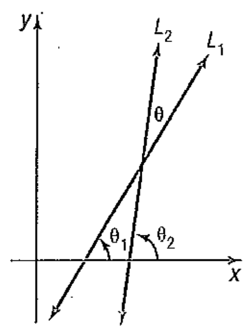
$$p(t) = V_m I_m \sin(\omega t)\sin(\omega t - \phi)$$

Source HyperPhysics, hosted by Georgia State University

107. **Geometry: Angle between Two Lines** Let L_1 and L_2 denote two nonvertical intersecting lines, and let θ denote the acute angle between L_1 and L_2 (see the figure). Show that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2, respectively. [**Hint:** Use the facts that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.]



108. If $\alpha + \beta + \gamma = 180°$ and

$$\cot\theta = \cot\alpha + \cot\beta + \cot\gamma, \quad 0 < \theta < 90°$$

show that

$$\sin^3\theta = \sin(\alpha - \theta)\sin(\beta - \theta)\sin(\gamma - \theta)$$

109. If $\tan\alpha = x + 1$ and $\tan\beta = x - 1$, show that

$$2\cot(\alpha - \beta) = x^2$$

Discussion and Writing

110. Discuss the following derivation:

$$\tan\left(\theta + \frac{\pi}{2}\right) = \frac{\tan\theta + \tan\frac{\pi}{2}}{1 - \tan\theta\tan\frac{\pi}{2}} = \frac{\frac{\tan\theta}{\tan\frac{\pi}{2}} + 1}{\frac{1}{\tan\frac{\pi}{2}} - \tan\theta} = \frac{0 + 1}{0 - \tan\theta} = \frac{1}{-\tan\theta} = -\cot\theta$$

Can you justify each step?

111. Explain why formula (7) cannot be used to show that

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

Establish this identity by using formulas (3a) and (3b).

Retain Your Knowledge

Problems 112–115 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

112. Determine the points of intersection of the graphs of $f(x) = x^2 + 5x + 1$ and $g(x) = -2x^2 - 11x - 4$ by solving $f(x) = g(x)$.

113. Convert $\frac{17\pi}{6}$ to degrees.

114. Find the area of the sector of a circle of radius 6 centimeters formed by an angle of 45°. Give both the exact area and an approximation rounded to two decimal places.

115. Given $\tan\theta = -2$, $270° < \theta < 360°$, find the exact value of the remaining five trigonometric functions.

'Are You Prepared?' Answers

1. 5 2. $-\frac{3}{5}$ 3. (a) $\frac{\sqrt{2}}{4}$ (b) $\frac{1}{2}$ 4. $-\frac{3}{5}$

6.6 Double-angle and Half-angle Formulas

OBJECTIVES
1 Use Double-angle Formulas to Find Exact Values (p. 512)
2 Use Double-angle Formulas to Establish Identities (p. 512)
3 Use Half-angle Formulas to Find Exact Values (p. 515)

In this section, formulas for $\sin(2\theta)$, $\cos(2\theta)$, $\sin\left(\frac{1}{2}\theta\right)$, and $\cos\left(\frac{1}{2}\theta\right)$ are established in terms of $\sin\theta$ and $\cos\theta$. They are derived using the sum formulas.

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta \quad \text{Let } \alpha = \beta = \theta$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$

and

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta \quad \text{Let } \alpha = \beta = \theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

An application of the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$ results in two other ways to express $\cos(2\theta)$.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

and

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

The following theorem summarizes the **Double-angle Formulas.**