PreCalc

6.6 Double-Angle and Half-Angle Formulas NOTES

Objectives:

1. Use Double-angle Formulas to find exact values.
2. Use Double-angle Formulas to Establish Identities.
3. Use Half-angle Formulas to find exact values.

We have already established the following formulas from section 6.5:

$\sin(\alpha+\beta) = sin\alpha\ cos\beta + cos\alpha\ sin\beta$ $\qquad$ $\cos(\alpha+\beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta$

Let $\theta = \alpha = \beta$, then....

$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$ $\qquad$ $\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$ $\qquad$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\rightarrow \sin^2\theta + \cos^2\theta = 1$

If we substitute the Pythagorean Identity, two additional ways to express $\cos(2\theta)$.

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$

Double-angle Formulas

$$\sin(2\theta) = 2sin\theta\ cos\theta \quad (1)$$

$$\cos(2\theta) = cos^2\theta - sin^2\theta \quad (2)$$

$$\cos(2\theta) = 1 - 2sin^2\theta \quad (3)$$

$$\cos(2\theta) = 2cos^2\theta - 1 \quad (4)$$

The above two formulas can be used to develop a formula for $tan^2\theta$.

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \quad *$$

Example. Establishing an Identity. Write an equilvalent expression for $cos^4\theta$ that does not involve any powers of sine or cosine greater than 1.

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$$\cos^4\theta = (\cos^2\theta)^2 = \left(\frac{1+\cos(2\theta)}{2}\right)^2$$

$$= \frac{1}{4}\left[1+\cos(2\theta)\right]^2$$

$$= \frac{1}{4}\left[1+2\cos(2\theta)+\cos^2(2\theta)\right]$$

$$= \frac{1}{4}+\frac{1}{2}\cos(2\theta)+\frac{1}{4}\left[\frac{1+\cos[2(2\theta)]}{2}\right]$$

$$= \frac{1}{4}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}(1+\cos(4\theta))$$

$$= \frac{1}{4}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}+\frac{1}{8}\cos(4\theta)$$

$$= \frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)$$

Example. Solving a Trig Eqn Using Identities. Solve the equation: $sin\theta\ cos\theta = -\frac{1}{2}$, $0 \le \theta < 2\pi$

$$\left[\sin\theta\cos\theta = -\frac{1}{2}\right]2$$

$$2\sin\theta\cos\theta = -1$$

$$\sin(2\theta) = -1$$

$$\left\{\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}\right\}$$

$$2\theta = \frac{3\pi}{2} + 2\pi k \quad [\div 2]$$

$$\theta = \frac{3\pi}{4} + \pi k$$

for $k = 1$

$$\frac{3\pi}{4} + \pi = \frac{3\pi}{4} + \frac{4\pi}{4} = \frac{7\pi}{4}$$

$\therefore$

$$\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\} \text{ on } [0, 2\pi)$$

Example. Find exact values using the Double-angle Formulas. If $sin\theta = \frac{3}{5}, \frac{\pi}{2} < \theta < \pi$, find the exact value of:

QII $(-,+)$

(a) $\sin(2\theta)$

(b) $\cos(2\theta)$

$\sin\theta = \frac{3}{5} = \frac{y}{r}$

let $y = 3, r = 5$

$x^2 + y^2 = r^2$

$x^2 + (3)^2 = 5^2$

$x^2 = 16$

$x = -4$ (b/c QII)

$\cos\theta = \frac{x}{r} = \frac{-4}{5}$

a) $\sin(2\theta) = 2\cos\theta\sin\theta$

$= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right)$

$= -\frac{24}{25}$

b) $\cos(2\theta) = 1 - 2\sin^2\theta$

$= 1 - 2\left(\frac{3}{5}\right)^2$

$= 1 - 2\left(\frac{9}{25}\right)$

$= 1 - \frac{18}{25}$

$= \frac{25}{25} - \frac{18}{25} = \frac{7}{25}$

Example. Develop a formula for a) $\tan(2\theta)$ in terms of $\tan\theta$.

Recall: $\tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha\, tan\beta}$

let $\theta = \alpha = \beta$

$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$

$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$ ☆

b) $\sin(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$

$\sin(3\theta) = \sin(2\theta + \theta)$

$= \sin(2\theta)\cos\theta + \sin\theta\cos(2\theta)$

$= 2\sin\theta\cos\theta\cos\theta + \sin\theta(\cos^2\theta - \sin^2\theta)$

$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$

$= 3\sin\theta\cos^2\theta - \sin^3\theta$

By rearranging Double-angle formulas (3) and (4), we get

$\cos(2\theta) = 1 - 2\sin^2\theta$

$+2\sin^2\theta \quad +2\sin^2\theta$

$\cos(2\theta) + 2\sin^2\theta = 1$

$-\cos(2\theta) \quad -\cos(2\theta)$

$\frac{2\sin^2\theta}{2} = \frac{1 - \cos(2\theta)}{2}$

$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$ ☆

$\cos(2\theta) = 2\cos^2\theta - 1$

$+1 \quad +1$

$\frac{2\cos^2\theta}{2} = \frac{\cos(2\theta) + 1}{2}$

$\cos^2\theta = \frac{\cos(2\theta) + 1}{2}$ ☆

Half-angle Formulas

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \qquad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha} \qquad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

where the + or – sign is determined by the quadrant of the angle $\frac{\alpha}{2}$

<u>Example</u>. Find the exact values using Half-angle Formulas. If $\cos\alpha = -\frac{3}{5}, \pi < \alpha < \frac{3\pi}{2}$, find the exact value of:

TAN

(a) $\sin\frac{\alpha}{2}$ (b) $\cos\frac{\alpha}{2}$ (c) $\tan\frac{\alpha}{2}$

if $\pi < \alpha < \frac{3\pi}{2}$

then $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$ $\therefore$ $\frac{\alpha}{2}$ is in QII $(-,+)$

a) $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{\frac{5}{5}+\frac{3}{5}}{2}} = \sqrt{\frac{\frac{8}{5}}{\frac{2}{1}}} = \sqrt{\frac{8}{10}} = \sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \boxed{\frac{2\sqrt{5}}{5}}$

b) $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$ (QII) $= -\sqrt{\frac{1-\frac{3}{5}}{2}} = -\sqrt{\frac{\frac{5}{5}-\frac{3}{5}}{2}} = -\sqrt{\frac{\frac{2}{5}}{\frac{2}{1}}} = -\sqrt{\frac{2}{5}\cdot\frac{1}{2}} = -\frac{\sqrt{1}}{\sqrt{5}} = -\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \boxed{-\frac{\sqrt{5}}{5}}$

c) $\tan\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = -\sqrt{\frac{\frac{5}{5}+\frac{3}{5}}{\frac{5}{5}-\frac{3}{5}}} = -\sqrt{\frac{\frac{8}{5}}{\frac{2}{5}}} = -\sqrt{4} = \boxed{-2}$

Half-angle Formulas for tan

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$