Solution See Figure 38. With a speed of 15 miles per hour, the sailboat has gone 60 miles after 4 hours. The distance x of the sailboat from Key West is to be found, along with the angle θ that the sailboat should turn through to correct its course.

(a) To find x, use the Law of Cosines, since two sides and the included angle are known.

$$x^2 = 150^2 + 60^2 - 2(150)(60)\cos 20° \approx 9185.53$$
$$x \approx 95.8$$

The sailboat is about 96 miles from Key West.

(b) With all three sides of the triangle now known, use the Law of Cosines again to find the angle A opposite the side of length 150 miles.

$$150^2 = 96^2 + 60^2 - 2(96)(60)\cos A$$
$$9684 = -11{,}520 \cos A$$
$$\cos A \approx -0.8406$$
$$A \approx 147.2°$$

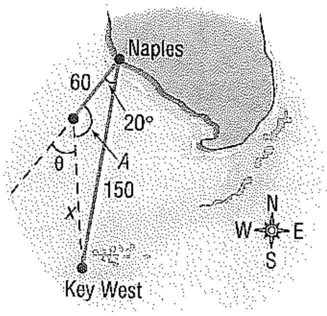
Hence

$$\theta = 180° - A \approx 180° - 147.2° = 32.8°$$

The sailboat should turn through an angle of about 33° to correct its course.

(c) The total length of the trip is now 60 + 96 = 156 miles. The extra 6 miles only requires about 0.4 hour, or 24 minutes, more if the speed of 15 miles per hour is maintained.

Figure 38



Now Work PROBLEM 45

Historical Feature

The Law of Sines was known vaguely long before it was explicitly stated by Nasir Eddin (about AD 1250). Ptolemy (about AD 150) was aware of it in a form using a chord function instead of the sine function. But it was first clearly stated in Europe by Regiomontanus, writing in 1464.

The Law of Cosines appears first in Euclid's *Elements* (Book II), but in a well-disguised form in which squares built on the sides of triangles are added and a rectangle representing the cosine term is subtracted. It was thus known to all mathematicians because of their familiarity with Euclid's work. An early modern form of the Law of Cosines, that for finding the angle when the sides are known, was stated by François Viète (in 1593).

The Law of Tangents (see Problem 61 of Section 7.2) has become obsolete. In the past it was used in place of the Law of Cosines, because the Law of Cosines was very inconvenient for calculation with logarithms or slide rules. Mixing addition and multiplication is now very easy on a calculator, however, and the Law of Tangents has been shelved along with the slide rule.

7.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Write the formula for the distance d from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$. (p. 3)

2. If θ is an acute angle, solve the equation $\cos\theta = \frac{\sqrt{2}}{2}$. (pp. 482–485)

Concepts and Vocabulary

3. If three sides of a triangle are given, the Law of ______ is used to solve the triangle.

4. If one side and two angles of a triangle are given, the Law of ______ is used to solve the triangle.

5. If two sides and the included angle of a triangle are given, the Law of ______ is used to solve the triangle.

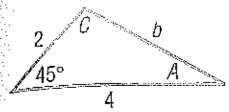
6. ***True or False*** Given only the three sides of a triangle, there is insufficient information to solve the triangle.

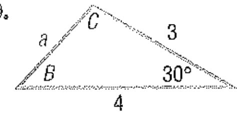
7. ***True or False*** Given two sides and the included angle, the first thing to do to solve the triangle is to use the Law of Sines.

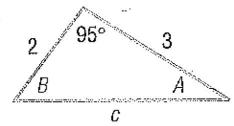
8. ***True or False*** A special case of the Law of Cosines is the Pythagorean Theorem.

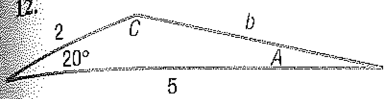
Skill Building

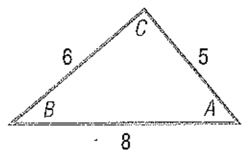
In Problems 9–16, solve each triangle.

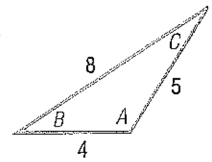
9.


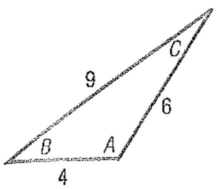
10.


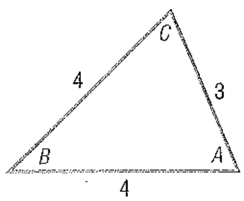
11.


12.


13.


14.


15.


16.


In Problems 17–32, solve each triangle.

17. $a = 3, \quad b = 4, \quad C = 40°$
18. $a = 2, \quad c = 1, \quad B = 10°$
19. $b = 1, \quad c = 3, \quad A = 80°$
20. $a = 6, \quad b = 4, \quad C = 60°$
21. $a = 3, \quad c = 2, \quad B = 110°$
22. $b = 4, \quad c = 1, \quad A = 120°$
23. $a = 2, \quad b = 2, \quad C = 50°$
24. $a = 3, \quad c = 2, \quad B = 90°$
25. $a = 12, \quad b = 13, \quad c = 5$
26. $a = 4, \quad b = 5, \quad c = 3$
27. $a = 2, \quad b = 2, \quad c = 2$
28. $a = 3, \quad b = 3, \quad c = 2$
29. $a = 5, \quad b = 8, \quad c = 9$
30. $a = 4, \quad b = 3, \quad c = 6$
31. $a = 10, \quad b = 8, \quad c = 5$
32. $a = 9, \quad b = 7, \quad c = 10$

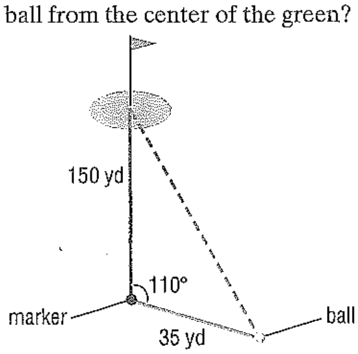
Mixed Practice

In Problems 33–42, solve each triangle using either the Law of Sines or the Law of Cosines.

33. $B = 20°, C = 75°, b = 5$
34. $A = 50°, B = 55°, c = 9$
35. $a = 6, b = 8, c = 9$
36. $a = 14, b = 7, A = 85°$
37. $B = 35°, C = 65°, a = 15$
38. $a = 4, c = 5, B = 55°$
39. $A = 10°, a = 3, b = 10$
40. $A = 65°, B = 72°, b = 7$
41. $b = 5, c = 12, A = 60°$
42. $a = 10, b = 10, c = 15$

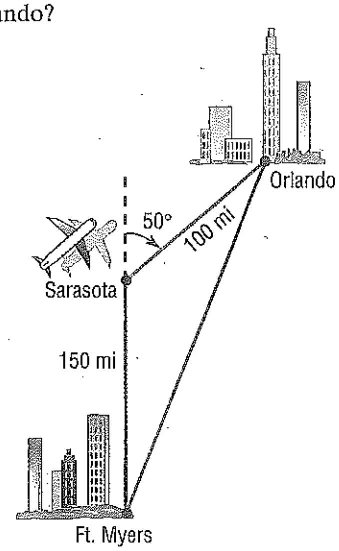
Applications and Extensions

43. **Distance to the Green** A golfer hits an errant tee shot that lands in the rough. A marker in the center of the fairway is 150 yards from the center of the green. While standing on the marker and facing the green, the golfer turns 110° toward his ball. He then paces off 35 yards to his ball. See the figure. How far is the ball from the center of the green?

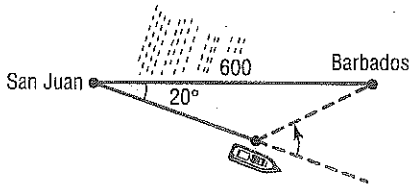


44. **Navigation** An airplane flies due north from Ft. Myers to Sarasota, a distance of 150 miles, and then turns through an angle of 50° and flies to Orlando, a distance of 100 miles. See the figure.

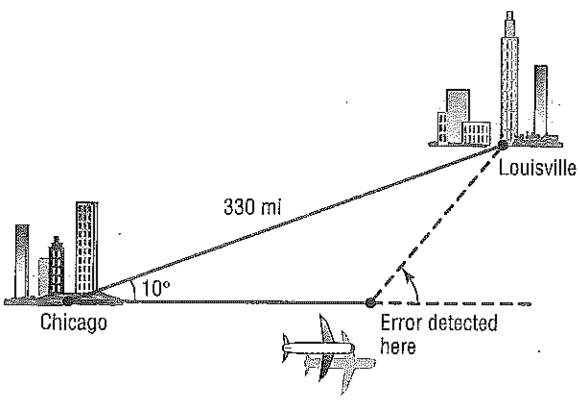
(a) How far is it directly from Ft. Myers to Orlando?
(b) What bearing should the pilot use to fly directly from Ft. Myers to Orlando?



45. **Avoiding a Tropical Storm** A cruise ship maintains an average speed of 15 knots in going from San Juan, Puerto Rico, to Barbados, West Indies, a distance of 600 nautical miles. To avoid a tropical storm, the captain heads out of San Juan in a direction of 20° off a direct heading to Barbados. The captain maintains the 15-knot speed for 10 hours, after which time the path to Barbados becomes clear of storms.
   (a) Through what angle should the captain turn to head directly to Barbados?
   (b) Once the turn is made, how long will it be before the ship reaches Barbados if the same 15-knot speed is maintained?



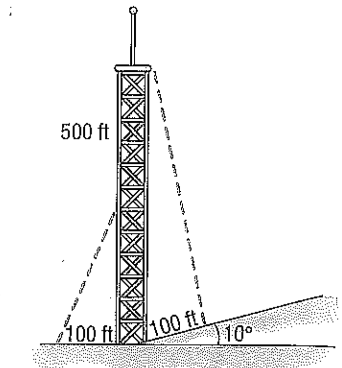
46. **Revising a Flight Plan** In attempting to fly from Chicago to Louisville, a distance of 330 miles, a pilot inadvertently took a course that was 10° in error, as indicated in the figure.
   (a) If the aircraft maintains an average speed of 220 miles per hour, and if the error in direction is discovered after 15 minutes, through what angle should the pilot turn to head toward Louisville?
   (b) What new average speed should the pilot maintain so that the total time of the trip is 90 minutes?



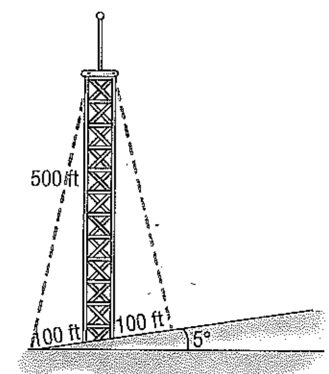
47. **Major League Baseball Field** A major league baseball diamond is actually a square 90 feet on a side. The pitching rubber is located 60.5 feet from home plate on a line joining home plate and second base.
   (a) How far is it from the pitching rubber to first base?
   (b) How far is it from the pitching rubber to second base?
   (c) If a pitcher faces home plate, through what angle does he need to turn to face first base?

48. **Little League Baseball Field** According to Little League baseball official regulations, the diamond is a square 60 feet on a side. The pitching rubber is located 46 feet from home plate on a line joining home plate and second base.
   (a) How far is it from the pitching rubber to first base?
   (b) How far is it from the pitching rubber to second base?
   (c) If a pitcher faces home plate, through what angle does he need to turn to face first base?

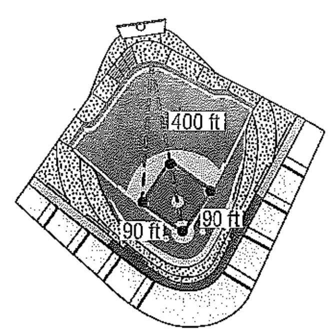
49. **Finding the Length of a Guy Wire** The height of a radio tower is 500 feet, and the ground on one side of the tower slopes upward at an angle of 10° (see the figure).
   (a) How long should a guy wire be if it is to connect to the top of the tower and be secured at a point on the sloped side 100 feet from the base of the tower?
   (b) How long should a second guy wire be if it is to connect to the middle of the tower and be secured at a point 100 feet from the base on the flat side?



50. **Finding the Length of a Guy Wire** A radio tower 500 feet high is located on the side of a hill with an inclination to the horizontal of 5°. See the figure. How long should two guy wires be if they are to connect to the top of the tower and be secured at two points 100 feet directly above and directly below the base of the tower?



51. **Wrigley Field, Home of the Chicago Cubs** The distance from home plate to the fence in dead center in Wrigley Field is 400 feet (see the figure). How far is it from the fence in dead center to third base?



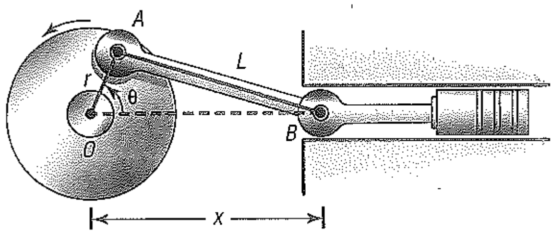
52. **Little League Baseball** The distance from home plate to the fence in dead center at the Oak Lawn Little League field is 280 feet. How far is it from the fence in dead center to third base?
[**Hint:** The distance between the bases in Little League is 60 feet.]

53. **Building a Swing Set** Clint is building a wooden swing set for his children. Each supporting end of the swing set is to be an A-frame constructed with two 10-foot-long 4-by-4s joined at a 45° angle. To prevent the swing set from tipping over, Clint wants to secure the base of each A-frame to concrete footings. How far apart should the footings for each A-frame be?

54. **Rods and Pistons** Rod OA rotates about the fixed point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston. See the figure. Show that the distance x between point O and point B is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$

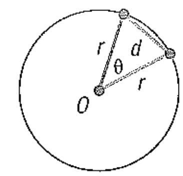
where θ is the angle of rotation of rod OA.



55. **Geometry** Show that the length d of a chord of a circle of radius r is given by the formula

$$d = 2r\sin\frac{\theta}{2}$$

where θ is the central angle formed by the radii to the ends of the chord. See the figure. Use this result to derive the fact that $\sin\theta < \theta$, where $\theta > 0$ is measured in radians.



56. For any triangle, show that

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

where $s = \frac{1}{2}(a + b + c)$.

[**Hint:** Use a Half-angle Formula and the Law of Cosines.]

57. For any triangle, show that

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

where $s = \frac{1}{2}(a + b + c)$.

58. Use the Law of Cosines to prove the identity

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Discussion and Writing

59. What do you do first if you are asked to solve a triangle and are given two sides and the included angle?
60. What do you do first if you are asked to solve a triangle and are given three sides?
61. Make up an applied problem that requires using the Law of Cosines.
62. Write down your strategy for solving an oblique triangle.
63. State the Law of Cosines in words.

Retain Your Knowledge

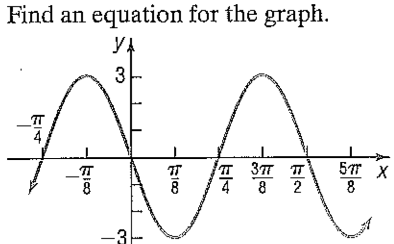
Problems 64–67 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

64. Graph: $R(x) = \dfrac{2x + 1}{x - 3}$

65. Solve $4^x = 3^{x+1}$. If the solution is irrational, express it both in exact form and as a decimal rounded to three places.

66. Given $\tan\theta = -\dfrac{2\sqrt{6}}{5}$ and $\cos\theta = -\dfrac{5}{7}$, find the exact value of each of the four remaining trigonometric functions.

67. Find an equation for the graph.



'Are You Prepared?' Answers

1. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. $\theta = 45°$ or $\frac{\pi}{4}$

7.4 Area of a Triangle

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Geometry Essentials (Appendix A, Section A.2, pp. A15–A16)

Now Work the 'Are You Prepared?' problem on page 564.

OBJECTIVES 1 Find the Area of SAS Triangles (p. 562)
2 Find the Area of SSS Triangles (p. 563)