1 Factor the Difference of Two Squares and the Sum and Difference of Two Cubes

When you factor a polynomial, first check for common monomial factors. Then see whether you can use one of the special formulas discussed in the previous section.

Difference of Two Squares	$a^2 - b^2 = (a - b)(a + b)$
Perfect Squares	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$
Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$x^n - m$, where a is a constant. I

Steps for Factoring $Ax^2 + Bx + C$, when $A \neq 1$ and A, B, and C Have No Common Factors

STEP 1: Find the value of AC.

STEP 2: Find integers whose product is AC that add up to B. That is, find a and b such that $ab = AC$ and $a + b = B$.

STEP 3: Write $Ax^2 + Bx + C = Ax^2 + ax + bx + C$.

STEP 4: Factor this last expression by grouping.

EXAMPLE 2 Factoring the Difference of Two Squares

Factor completely: $x^2 - 4$

Solution Note that $x^2 - 4$ is the difference of two squares, x^2 and 2^2.

$$x^2 - 4 = (x - 2)(x + 2)$$

EXAMPLE 3 Factoring the Difference of Two Cubes

Factor completely: $x^3 - 1$

Solution Because $x^3 - 1$ is the difference of two cubes, x^3 and 1^3,

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

EXAMPLE 4 Factoring the Sum of Two Cubes

Factor completely: $x^3 + 8$

Solution Because $x^3 + 8$ is the sum of two cubes, x^3 and 2^3,

$$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$

A.3 Assess Your Understanding

Concepts and Vocabulary

1. The polynomial $3x^4 - 2x^3 + 13x^2 - 5$ is of degree _____. The leading coefficient is _____.
2. $(x^2 - 4)(x^2 + 4) =$ _____.
3. $(x - 2)(x^2 + 2x + 4) =$ _____.
4. ***True or False*** $4x^{-2}$ is a monomial of degree -2.
5. ***True or False*** The degree of the product of two nonzero polynomials equals the sum of their degrees.
6. ***True or False*** $(x + a)(x^2 + ax + a) = x^3 + a^3$.

Skill Building

In Problems 7–16, tell whether the expression is a monomial. If it is, name the variable(s) and the coefficient and give the degree of the monomial. If it is not a monomial, state why not.

7. $2x^3$
8. $-4x^2$
9. $\frac{8}{x}$
10. $-2x^{-3}$
11. $-2xy^2$
12. $5x^2y^3$
13. $\frac{8x}{y}$
14. $-\frac{2x^2}{y^3}$
15. $x^2 + y^2$
16. $3x^2 + 4$

In Problems 17–26, tell whether the expression is a polynomial. If it is, give its degree. If it is not, state why not.

17. $3x^2 - 5$
18. $1 - 4x$
19. 5
20. $-\pi$
21. $3x^2 - \frac{5}{x}$
22. $\frac{3}{x} + 2$
23. $2y^3 - \sqrt{2}$
24. $10z^2 + z$
25. $\frac{x^2 + 5}{x^3 - 1}$
26. $\frac{3x^3 + 2x - 1}{x^2 + x + 1}$

In Problems 27–46, add, subtract, or multiply, as indicated. Express your answer as a single polynomial in standard form.

27. $(x^2 + 4x + 5) + (3x - 3)$
28. $(x^3 + 3x^2 + 2) + (x^2 - 4x + 4)$
29. $(x^3 - 2x^2 + 5x + 10) - (2x^2 - 4x + 3)$
30. $(x^2 - 3x - 4) - (x^3 - 3x^2 + x + 5)$
31. $(6x^5 + x^3 + x) + (5x^4 - x^3 + 3x^2)$
32. $(10x^5 - 8x^2) + (3x^3 - 2x^2 + 6)$
33. $(x^2 - 3x + 1) + 2(3x^2 + x - 4)$
34. $-2(x^2 + x + 1) + (-5x^2 - x + 2)$
35. $6(x^3 + x^2 - 3) - 4(2x^3 - 3x^2)$
36. $8(4x^3 - 3x^2 - 1) - 6(4x^3 + 8x - 2)$
37. $(x^2 - x + 2) + (2x^2 - 3x + 5) - (x^2 + 1)$
38. $(x^2 + 1) - (4x^2 + 5) + (x^2 + x - 2)$
39. $9(y^2 - 3y + 4) - 6(1 - y^2)$
40. $8(1 - y^3) + 4(1 + y + y^2 + y^3)$
41. $x(x^2 + x - 4)$
42. $4x^2(x^3 - x + 2)$
43. $-2x^2(4x^3 + 5)$
44. $5x^3(3x - 4)$
45. $(x + 1)(x^2 + 2x - 4)$
46. $(2x - 3)(x^2 + x + 1)$

In Problems 47–64, multiply the polynomials using the FOIL method. Express your answer as a single polynomial in standard form.

47. $(x + 2)(x + 4)$
48. $(x + 3)(x + 5)$
49. $(2x + 5)(x + 2)$
50. $(3x + 1)(2x + 1)$
51. $(x - 4)(x + 2)$
52. $(x + 4)(x - 2)$
53. $(x - 3)(x - 2)$
54. $(x - 5)(x - 1)$
55. $(2x + 3)(x - 2)$
56. $(2x - 4)(3x + 1)$
57. $(-2x + 3)(x - 4)$
58. $(-3x - 1)(x + 1)$
59. $(-x - 2)(-2x - 4)$
60. $(-2x - 3)(3 - x)$
61. $(x - 2y)(x + y)$
62. $(2x + 3y)(x - y)$
63. $(-2x - 3y)(3x + 2y)$
64. $(x - 3y)(-2x + y)$

In Problems 65–88, multiply the polynomials using the special product formulas. Express your answer as a single polynomial in standard form.

65. $(x - 7)(x + 7)$
66. $(x - 1)(x + 1)$
67. $(2x + 3)(2x - 3)$
68. $(3x + 2)(3x - 2)$
69. $(x + 4)^2$
70. $(x + 5)^2$
71. $(x - 4)^2$
72. $(x - 5)^2$
73. $(3x + 4)(3x - 4)$
74. $(5x - 3)(5x + 3)$
75. $(2x - 3)^2$
76. $(3x - 4)^2$
77. $(x + y)(x - y)$
78. $(x + 3y)(x - 3y)$
79. $(3x + y)(3x - y)$
80. $(3x + 4y)(3x - 4y)$
81. $(x + y)^2$
82. $(x - y)^2$
83. $(x - 2y)^2$
84. $(2x + 3y)^2$
85. $(x - 2)^3$
86. $(x + 1)^3$
87. $(2x + 1)^3$
88. $(3x - 2)^3$

In Problems 89–104, find the quotient and the remainder. Check your work by verifying that

$$(Quotient)(Divisor) + Remainder = Dividend$$

89. $4x^3 - 3x^2 + x + 1$ divided by $x + 2$
90. $3x^3 - x^2 + x - 2$ divided by $x + 2$
91. $4x^3 - 3x^2 + x + 1$ divided by x^2
92. $3x^3 - x^2 + x - 2$ divided by x^2
93. $5x^4 - 3x^2 + x + 1$ divided by $x^2 + 2$
94. $5x^4 - x^2 + x - 2$ divided by $x^2 + 2$
95. $4x^5 - 3x^2 + x + 1$ divided by $2x^3 - 1$
96. $3x^5 - x^2 + x - 2$ divided by $3x^3 - 1$
97. $2x^4 - 3x^3 + x + 1$ divided by $2x^2 + x + 1$
98. $3x^4 - x^3 + x - 2$ divided by $3x^2 + x + 1$
99. $-4x^3 + x^2 - 4$ divided by $x - 1$
100. $-3x^4 - 2x - 1$ divided by $x - 1$
101. $1 - x^2 + x^4$ divided by $x^2 + x + 1$
102. $1 - x^2 + x^4$ divided by $x^2 - x + 1$
103. $x^3 - a^3$ divided by $x - a$
104. $x^5 - a^5$ divided by $x - a$

Completing the Square

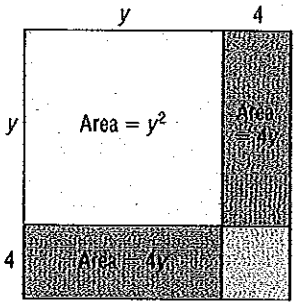
Identify the coefficient of the first-degree term. Multiply this coefficient by $\frac{1}{2}$ and then square the result. That is, determine the value of b in $x^2 + bx$ and compute $\left(\frac{1}{2}b\right)^2$.

EXAMPLE 19 Completing the Square

Determine the number that must be added to each expression to complete the square. Then factor the expression.

Start	Add	Result	Factored Form
$y^2 + 8y$	$\left(\frac{1}{2}\cdot 8\right)^2 = 16$	$y^2 + 8y + 16$	$(y + 4)^2$
$x^2 + 12x$	$\left(\frac{1}{2}\cdot 12\right)^2 = 36$	$x^2 + 12x + 36$	$(x + 6)^2$
$a^2 - 20a$	$\left(\frac{1}{2}\cdot(-20)\right)^2 = 100$	$a^2 - 20a + 100$	$(a - 10)^2$
$p^2 - 5p$	$\left(\frac{1}{2}\cdot(-5)\right)^2 = \frac{25}{4}$	$p^2 - 5p + \frac{25}{4}$	$\left(p - \frac{5}{2}\right)^2$

Figure 24



Note that the factored form of a perfect square is either

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \quad \text{or} \quad x^2 - bx + \left(\frac{b}{2}\right)^2 = \left(x - \frac{b}{2}\right)^2$$

Now Work PROBLEM 69

Are you wondering why making an expression a perfect square is called "completing the square"? Look at the square in Figure 24. Its area is $(y + 4)^2$. The yellow area is y^2 and each orange area is $4y$ (for a total area of $8y$). The sum of these areas is $y^2 + 8y$. To complete the square, we need to add the area of the green region: $4 \cdot 4 = 16$. As a result, $y^2 + 8y + 16 = (y + 4)^2$.

A.4 Assess Your Understanding

Concepts and Vocabulary

1. If factored completely, $3x^3 - 12x =$ _____.
2. If a polynomial cannot be written as the product of two other polynomials (excluding 1 and -1), then the polynomial is said to be _____.
3. *True or False* The polynomial $x^2 + 4$ is prime.
4. *True or False* $3x^3 - 2x^2 - 6x + 4 = (3x - 2)(x^2 + 2)$.

Skill Building

In Problems 5–14, factor each polynomial by removing the common monomial factor.

5. $3x + 6$
6. $7x - 14$
7. $ax^2 + a$
8. $ax - a$
9. $x^3 + x^2 + x$
10. $x^3 - x^2 + x$
11. $2x^2 - 2x$
12. $3x^2 - 3x$
13. $3x^2y - 6xy^2 + 12xy$
14. $60x^2y - 48xy^2 + 72x^3y$

In Problems 15–22, factor the difference of two squares.

15. $x^2 - 1$
16. $x^2 - 4$
17. $4x^2 - 1$
18. $9x^2 - 1$
19. $x^2 - 16$
20. $x^2 - 25$
21. $25x^2 - 4$
22. $36x^2 - 9$

In Problems 23–32, factor the perfect squares.

23. $x^2 + 2x + 1$
24. $x^2 - 4x + 4$
25. $x^2 + 4x + 4$
26. $x^2 - 2x + 1$
27. $x^2 - 10x + 25$
28. $x^2 + 10x + 25$
29. $4x^2 + 4x + 1$
30. $9x^2 + 6x + 1$
31. $16x^2 + 8x + 1$
32. $25x^2 + 10x + 1$

In Problems 33–38, factor the sum or difference of two cubes.

33. $x^3 - 27$
34. $x^3 + 125$
35. $x^3 + 27$
36. $27 - 8x^3$
37. $8x^3 + 27$
38. $64 - 27x^3$

In Problems 39–50, factor each polynomial.

39. $x^2 + 5x + 6$
40. $x^2 + 6x + 8$
41. $x^2 + 7x + 6$
42. $x^2 + 9x + 8$
43. $x^2 + 7x + 10$
44. $x^2 + 11x + 10$
45. $x^2 - 10x + 16$
46. $x^2 - 17x + 16$
47. $x^2 - 7x - 8$
48. $x^2 - 2x - 8$
49. $x^2 + 7x - 8$
50. $x^2 + 2x - 8$

In Problems 51–56, factor by grouping.

51. $2x^2 + 4x + 3x + 6$
52. $3x^2 - 3x + 2x - 2$
53. $2x^2 - 4x + x - 2$
54. $3x^2 + 6x - x - 2$
55. $6x^2 + 9x + 4x + 6$
56. $9x^2 - 6x + 3x - 2$

In Problems 57–68, factor each polynomial.

57. $3x^2 + 4x + 1$
58. $2x^2 + 3x + 1$
59. $2z^2 + 5z + 3$
60. $6z^2 + 5z + 1$
61. $3x^2 + 2x - 8$
62. $3x^2 + 10x + 8$
63. $3x^2 - 2x - 8$
64. $3x^2 - 10x + 8$
65. $3x^2 + 14x + 8$
66. $3x^2 - 14x + 8$
67. $3x^2 + 10x - 8$
68. $3x^2 - 10x - 8$

In Problems 69–74, determine the number that should be added to complete the square of each expression. Then factor each expression.

69. $x^2 + 10x$
70. $p^2 + 14p$
71. $y^2 - 6y$
72. $x^2 - 4x$
73. $x^2 - \frac{1}{2}x$
74. $x^2 + \frac{1}{3}x$

Mixed Practice

In Problems 75–122, factor each polynomial completely. If the polynomial cannot be factored, say it is prime.

75. $x^2 - 36$
76. $x^2 - 9$
77. $2 - 8x^2$
78. $3 - 27x^2$
79. $x^2 + 11x + 10$
80. $x^2 + 5x + 4$
81. $x^2 - 10x + 21$
82. $x^2 - 6x + 8$
83. $4x^2 - 8x + 32$
84. $3x^2 - 12x + 15$
85. $x^2 + 4x + 16$
86. $x^2 + 12x + 36$
87. $15 + 2x - x^2$
88. $14 + 6x - x^2$
89. $3x^2 - 12x - 36$
90. $x^3 + 8x^2 - 20x$
91. $y^4 + 11y^3 + 30y^2$
92. $3y^3 - 18y^2 - 48y$
93. $4x^2 + 12x + 9$
94. $9x^2 - 12x + 4$
95. $6x^2 + 8x + 2$
96. $8x^2 + 6x - 2$
97. $x^4 - 81$
98. $x^4 - 1$
99. $x^6 - 2x^3 + 1$
100. $x^6 + 2x^3 + 1$
101. $x^7 - x^5$
102. $x^8 - x^5$
103. $16x^2 + 24x + 9$
104. $9x^2 - 24x + 16$
105. $5 + 16x - 16x^2$
106. $5 + 11x - 16x^2$
107. $4y^2 - 16y + 15$
108. $9y^2 + 9y - 4$
109. $1 - 8x^2 - 9x^4$
110. $4 - 14x^2 - 8x^4$
111. $x(x + 3) - 6(x + 3)$
112. $5(3x - 7) + x(3x - 7)$
113. $(x + 2)^2 - 5(x + 2)$
114. $(x - 1)^2 - 2(x - 1)$
115. $(3x - 2)^3 - 27$
116. $(5x + 1)^3 - 1$
117. $3(x^2 + 10x + 25) - 4(x + 5)$
118. $7(x^2 - 6x + 9) + 5(x - 3)$
119. $x^3 + 2x^2 - x - 2$
120. $x^3 - 3x^2 - x + 3$
121. $x^4 - x^3 + x - 1$
122. $x^4 + x^3 + x + 1$

Applications and Extensions

In Problems 123–132, expressions that occur in calculus are given. Factor each expression completely.

123. $2(3x + 4)^2 + (2x + 3) \cdot 2(3x + 4) \cdot 3$
124. $5(2x + 1)^2 + (5x - 6) \cdot 2(2x + 1) \cdot 2$
125. $2x(2x + 5) + x^2 \cdot 2$
126. $3x^2(8x - 3) + x^3 \cdot 8$
127. $2(x + 3)(x - 2)^3 + (x + 3)^2 \cdot 3(x - 2)^2$
128. $4(x + 5)^3(x - 1)^2 + (x + 5)^4 \cdot 2(x - 1)$